

# 1

## Integrali i pacaktuar

### 1. Integralet tabelare

Duke zbatuar tabelën e integrale të zgjidhen integralet:

1.  $\int (x^2 + 3x + 4)dx.$

2.  $\int \frac{x-3}{x^4} dx.$

3.  $\int \frac{dx}{\sqrt[3]{x^2}}.$

4.  $\int \sqrt{x \cdot \sqrt[3]{x}} dx.$

5.  $\int \left(1 - \frac{1}{x}\right) \sqrt{x\sqrt{x}} dx.$

6.  $\int \frac{16-x}{4+\sqrt{x}} dx.$

7.  $\int \frac{8x^2-2}{2x+1} dx.$

**Zgjidhja.**

1.  $\int (x^2 + 3x + 4)dx = \int x^2 dx + \int 3x dx + \int 4dx = \frac{x^3}{3} + 3 \int x dx + 4 \int dx$   
 $= \frac{x^3}{3} + 3 \cdot \frac{x^2}{2} + 4x + C.$

2.  $\int \frac{x-3}{x^4} dx = \int \frac{x}{x^4} dx - \int \frac{3}{x^4} dx = \int \frac{dx}{x^3} - 3 \int \frac{dx}{x^4} = \int x^{-3} dx - 3 \int x^{-4} dx$

$$= \frac{x^{-3+1}}{-3+1} - 3 \frac{x^{-4+1}}{-4+1} = -\frac{1}{2x^2} + \frac{1}{x^3} + C.$$

$$3. \int \frac{dx}{\sqrt[3]{x^2}} = \int \frac{dx}{x^{\frac{2}{3}}} = \int x^{-\frac{2}{3}} dx = \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} = \frac{x^{\frac{1}{3}}}{\frac{1}{3}} = 3\sqrt[3]{x} + C.$$

$$4. \int \sqrt{x} \cdot \sqrt[3]{x} dx = \int \sqrt{x} \cdot x^{\frac{1}{3}} dx = \int \sqrt{x^{\frac{4}{3}}} dx = \int x^{\frac{2}{3}} dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} = \frac{3}{5} \sqrt[3]{x^5} + C.$$

$$5. \int \left(1 - \frac{1}{x}\right) \sqrt{x} \sqrt{x} dx = \int \left(1 - \frac{1}{x}\right) x^{\frac{1}{2}} x^{\frac{1}{2}} dx = \int \left(1 - \frac{1}{x}\right) x^1 dx$$

$$= \int x^1 dx - \int x^{-1} \cdot x^1 dx = \frac{x^{1+1}}{\frac{1}{1}+1} - \int x^{-1+\frac{3}{4}} dx$$

$$= \frac{x^2}{2} - \frac{x^{\frac{3}{4}}}{\frac{3}{4}} = \frac{2}{7} x^{\frac{7}{4}} - \frac{4}{3} x^{\frac{3}{4}} + C.$$

### 6. Mënyra e parë.

$$\int \frac{16-x}{4+\sqrt{x}} dx = \int \frac{4^2 - (\sqrt{x})^2}{4+\sqrt{x}} dx = \int \frac{(4-\sqrt{x})(4+\sqrt{x})}{4+\sqrt{x}} dx = \int (4-\sqrt{x}) dx$$

$$= 4 \int dx - \int x^{\frac{1}{2}} dx = 4x - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = 4x - \frac{2}{3} \sqrt{x^3} + C.$$

### Mënyra e dytë.

Kryjemë racionalizimin e emëruesit:

$$\int \frac{16-x}{4+\sqrt{x}} dx = \int \frac{16-x}{4+\sqrt{x}} \cdot \frac{4-\sqrt{x}}{4-\sqrt{x}} dx = \int \frac{(16-x)(4-\sqrt{x})}{16-x} dx = \int (4-\sqrt{x}) dx$$

$$= 4x - \frac{2}{3} \sqrt{x^3} + C.$$

$$\begin{aligned}
 7. \int \frac{8x^2 - 2}{2x + 1} dx &= \int \frac{2(4x^2 - 1)}{2x + 1} dx = 2 \int \frac{(2x + 1)(2x - 1)}{(2x + 1)} dx = 2 \int (2x - 1) dx \\
 &= 2 \left( \int x dx - \int dx \right) = 4 \frac{x^2}{2} - 2x = 2x^2 - 2x + C.
 \end{aligned}$$

### ***Detyra plotësuese***

Të njehsohen integralet:

- |                                                                                            |                                                      |                                 |
|--------------------------------------------------------------------------------------------|------------------------------------------------------|---------------------------------|
| 1. $\int (x^3 - x + 2) dx.$                                                                | 2. $\int \sqrt{x}(x + 1) dx.$                        | 3. $\int x^4 \sqrt{x^3} dx.$    |
| 4. $\int \frac{dx}{x^4}.$                                                                  | 5. $\int \frac{dx}{\sqrt[4]{x}}.$                    | 6. $\int \frac{dx}{x\sqrt{x}}.$ |
| 7. $\int (x + 1)^3 dx.$                                                                    | 8. $\int (x^2 + 1)^2 dx.$                            | 9. $\int x^3(1 - x^2) dx.$      |
| 10. $\int \left( x^2 + x\sqrt{x} + x^2 \cdot \sqrt[3]{x} + \frac{1}{\sqrt{x}} \right) dx.$ | 11. $\int \frac{\sqrt{x} - 2x + 3}{\sqrt[3]{x}} dx.$ |                                 |
| 12. $\int \frac{x^4 - 1}{x^2} dx.$                                                         | 13. $\int \frac{(x + 1)(x^2 + 1)}{x^3} dx.$          |                                 |
| 14. $\int \frac{9 - \sqrt{x}}{3 - \sqrt{x}} dx.$                                           | 15. $\int \frac{9x^2 - 3}{\sqrt{3x + 1}} dx.$        |                                 |

Të njehsohen integralet:

8.  $\int e^x \left( 1 + \frac{e^{-x}}{x} \right) dx.$
9.  $\int (e^{2+x} - 2^x) dx.$
10.  $\int (e^x - x^e) dx.$
11.  $\int \frac{e^{3x} + 8}{e^x + 2} dx.$
12.  $\int (2^x - 3^x) dx.$
13.  $\int 2^x \cdot 3^{-x} dx.$

$$14. \int (3^x + 4^x)^2 dx.$$

$$15. \int \frac{3^{x+1} - 5^{x-1}}{15^x} dx.$$

**Zgjidhja.**

$$8. \int e^x \left( 1 + \frac{e^{-x}}{x} \right) dx = \int e^x dx + \int \frac{dx}{x} = e^x + \ln |x| + C.$$

$$9. \int (e^{2+x} - 2^x) dx = \int (e^{2+x} - 2^x) dx = \int e^{2+x} dx - \int 2^x dx = \int e^2 e^x dx - \frac{2^x}{\ln 2} \\ = e^2 \int e^x dx - \frac{2^x}{\ln 2} = e^2 e^x - \frac{2^x}{\ln 2} = e^{2+x} - \frac{2^x}{\ln 2} + C.$$

$$10. \int (e^x - x^e) dx = \int e^x dx - \int x^e dx = e^x - \frac{x^{e+1}}{e+1} + C.$$

$$11. \int \frac{e^{3x} + 8}{e^x + 2} dx = \int \frac{(e^x)^3 + 2^3}{e^x + 2} dx = \int \frac{(e^x + 2)(e^{2x} - 2e^x + 4)}{e^x + 2} \\ = \int e^{2x} dx - 2 \int e^x dx + 4 \int dx = \int (e^2)^x dx - 2e^x + 4x \\ = \frac{(e^2)^x}{\ln e^2} - 2e^x + 4x = \frac{e^{2x}}{2 \ln e} - 2e^x + 4x = \frac{e^{2x}}{2} - 2e^x + 4x + C.$$

$$12. \int (2^x - 3^x) dx = \int 2^x dx - \int 3^x dx = \frac{2^x}{\ln 2} - \frac{3^x}{\ln 3} + C.$$

$$13. \int 2^x \cdot 3^{-x} dx = \int 2^x (3^{-1})^x dx = \int \left( \frac{2}{3} \right)^x dx = \frac{\left( \frac{2}{3} \right)^x}{\ln \frac{2}{3}} + C.$$

$$14. \int (3^x + 4^x)^2 dx = \int (3^{2x} + 2 \cdot 3^x 4^x + 4^{2x}) dx = \int 9^x dx + 2 \int 12^x dx + \int 16^x dx \\ = \frac{9^x}{\ln 9} + 2 \frac{12^x}{\ln 12} + \frac{16^x}{\ln 16} + C.$$

$$15. \int \frac{3^{x+1} - 5^{x-1}}{15^x} dx = \int \frac{3 \cdot 3^x - 5^{-1} \cdot 5^x}{15^x} dx = 3 \int \left( \frac{1}{5} \right)^x dx - \frac{1}{5} \int \left( \frac{1}{3} \right)^x dx$$

$$= 3 \frac{\left(\frac{1}{5}\right)^x}{\ln \frac{1}{5}} - \frac{1}{5} \frac{\left(\frac{1}{3}\right)^x}{\ln \frac{1}{3}} = -\frac{3}{\ln 5} 5^{-x} + \frac{1}{5 \ln 5} 3^{-x} + C.$$

### *Detyra plotësuese*

Të njehsohen integralet:

16. $\int e^x \left(1 - \frac{e^{-x}}{x^2}\right) dx.$	17. $\int \left( e^{x+\frac{1}{2}} - \left(\frac{1}{2}\right)^{x+2} \right) dx.$
18. $\int (e^{2x} - x^{2e}) dx.$	19. $\int \frac{e^{4x} - 1}{e^{2x} + 1} dx.$
20. $\int \frac{e^{3x} - 27}{e^x - 3} dx.$	21. $\int 3^x \cdot 5^{-x} dx.$
22. $\int (2^x + 2^{-x}) dx.$	23. $\int (3^x - 3^{-x})^2 dx.$
24. $\int \frac{2^{x-1} - 3^{x+1}}{6^x} dx.$	25. $\int (e^{x+e} - x^{e-1}) dx.$

Të njehsohen integralet:

16.  $\int \frac{x^2}{x^2 + 1} dx.$
17.  $\int \frac{x^4}{1 + x^2} dx.$
18.  $\int \frac{(1+x)^2}{x(1+x^2)} dx.$
19.  $\int \frac{x^2 + 4}{x^2 + 1} dx.$
20.  $\int \frac{x^3 + x - 2}{x^2 + 1} dx.$
21.  $\int \left( 4e^x - \frac{5}{\sqrt{4 - 4x^2}} \right) dx.$

**Zgjidhja.****16. Mënyra e parë.**

Pjesëtojmë polinomit  $x^2$  me polinomin  $x^2 + 1$ . Merret:

$$\frac{x^2}{x^2 + 1} = 1 - \frac{1}{x^2 + 1}.$$

Pra,

$$\int \frac{x^2}{x^2 + 1} dx = \int \left( 1 - \frac{1}{x^2 + 1} \right) dx = \int dx - \int \frac{dx}{x^2 + 1} = x - \arctan x + C.$$

**Mënyra e dytë.**

$$\begin{aligned} \int \frac{x^2}{x^2 + 1} dx &= \int \frac{x^2 + 1 - 1}{x^2 + 1} dx = \int \left( \frac{x^2 + 1}{x^2 + 1} - \frac{1}{x^2 + 1} \right) dx \\ &= \int \left( 1 - \frac{1}{x^2 + 1} \right) dx = x - \arctan x + C. \end{aligned}$$

$$\begin{aligned} 17. \int \frac{x^4}{1 + x^2} dx &= \int \frac{x^4 - 1 + 1}{1 + x^2} dx = \int \frac{x^4 - 1}{1 + x^2} dx + \int \frac{1}{1 + x^2} dx \\ &= \int \frac{(x^2 - 1)(x^2 + 1)}{1 + x^2} dx + \arctan x = \int (x^2 - 1) dx + \arctan x \\ &= \frac{x^3}{3} - x + \arctan x + C. \end{aligned}$$

$$\begin{aligned} 18. \int \frac{(1 + x)^2}{x(1 + x^2)} dx &= \int \frac{1 + 2x + x^2}{x(1 + x^2)} dx = \int \frac{1 + x^2}{x(1 + x^2)} dx + \int \frac{2x}{x(1 + x^2)} dx \\ &= \int \frac{dx}{x} + 2 \int \frac{dx}{1 + x^2} = \ln |x| + 2 \arctan x + C. \end{aligned}$$

$$\begin{aligned} 19. \int \frac{x^2 + 4}{x^2 + 1} dx &= \int \frac{x^2 + 1 + 3}{x^2 + 1} dx = \int \left( 1 + \frac{3}{x^2 + 1} \right) dx = \int dx + 3 \int \frac{dx}{x^2 + 1} \\ &= x + 3 \arctan x + C. \end{aligned}$$

**20.** Pas pjesëtimit të polinomeve  $x^3 + x - 2$  me  $x^2 + 1$  merret:

$$\frac{x^3 + x - 2}{x^2 + 1} = x - \frac{2}{x^2 + 1}.$$

Kemi:

$$\int \frac{x^3 + x - 2}{x^2 + 1} dx = \int \left( x - \frac{2}{x^2 + 1} \right) dx = \frac{x^2}{2} - 2 \arctan x + C.$$

$$21. \int \left( 4e^x - \frac{5}{\sqrt{4 - 4x^2}} \right) dx = 4 \int e^x - 5 \int \frac{dx}{2\sqrt{1 - x^2}} = 4e^x - \frac{5}{2} \arcsin x + C.$$

### ***Detyra plotësuese***

Të njehsohen integralet:

$$26. \int \frac{3x^2 - 1}{x^2 + 1} dx.$$

$$27. \int \frac{x^4 - 1}{x^2 + 1} dx.$$

$$28. \int \frac{(1+x)^3 - x^2 - x^3}{x(1+x)} dx.$$

$$29. \int \frac{x^3 + x - 3}{x^2 + 1} dx.$$

$$30. \int \frac{x^8 - 1}{x^2 + 1} dx.$$

$$31. \int \left( 3^x - \frac{1}{\sqrt{\frac{1}{9} - \left(\frac{x}{3}\right)^2}} \right) dx.$$

Të njehsohen integralet:

$$22. \int (x - \sin x + 2 \cos x) dx.$$

$$23. \int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x} dx.$$

$$24. \int \frac{dx}{\sin^2 x \cdot \cos^2 x}.$$

$$25. \int \tan^2 x dx.$$

$$26. \int \cos^2 \frac{x}{2} dx.$$

$$27. \int \sqrt{1 - \sin 2x} dx.$$

### ***Zgjidhja.***

$$22. \int (x - \sin x + 2 \cos x) dx = \int x dx - \int \sin x dx + 2 \int \cos x dx$$

$$\begin{aligned}
 &= \frac{x^2}{2} - (-\cos x) + 2 \sin x + C \\
 &= \frac{x^2}{2} + \cos x + 2 \sin x + C.
 \end{aligned}$$

**23.** Zbatojmë formulën

$$\cos 2x = \cos^2 x - \sin^2 x.$$

Merret:

$$\begin{aligned}
 \int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x} dx &= \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cdot \cos^2 x} dx \\
 &= \int \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x} dx - \int \frac{\sin^2 x}{\sin^2 x \cdot \cos^2 x} dx \\
 &= \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} = -\cot x - \tan x + C.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{24.} \int \frac{dx}{\sin^2 x \cdot \cos^2 x} &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx = \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} \\
 &= \tan x - \cot x + C.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{25.} \int \tan^2 x dx &= \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1 - \cos^2 x}{\cos^2 x} dx = \int \frac{dx}{\cos^2 x} - \int dx \\
 &= \tan x - x + C.
 \end{aligned}$$

$$\mathbf{26.} \text{ Meqë } \cos^2 x = \frac{1 + \cos 2x}{2} \text{ atëherë } \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}.$$

D.m.th.

$$\int \cos^2 \frac{x}{2} dx = \int \frac{1 + \cos x}{2} dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos x dx = \frac{1}{2} x + \frac{1}{2} \sin x + C.$$

$$\begin{aligned}
 \mathbf{27.} \int \sqrt{1 - \sin 2x} dx &= \int \sqrt{\sin^2 x + \cos^2 x - 2 \sin x \cos x} dx \\
 &= \int \sqrt{(\sin x - \cos x)^2} = \int |\sin x - \cos x| dx \\
 &= \int (\sin x - \cos x) \cdot \operatorname{sgn}(\sin x - \cos x) dx \\
 &= (-\cos x - \sin x) \operatorname{sgn}(\sin x - \cos x)
 \end{aligned}$$

$$= -(\sin x + \cos x) \operatorname{sgn}(\sin x - \cos x)$$

$$= (\sin x + \cos x) \operatorname{sgn}(\cos x - \sin x).$$

### *Detyra plotësuese*

Të njehsohen integralet:

$$32. \int \left( \frac{x}{2} - 2 \sin x + 3 \cos x \right) dx.$$

$$33. \int \frac{\cos 2x}{\sin^2 2x} dx.$$

$$34. \int \frac{4}{\sin^2 2x} dx.$$

$$35. \int \cot^2 x dx.$$

$$36. \int \sin^2 \frac{x}{2} dx.$$

$$37. \int \left( \sin \frac{x}{2} + \cos \frac{x}{2} \right)^2 dx.$$

$$38. \int \frac{1 - \sin^3 x}{\sin^3 x} dx.$$