

2. Integrimi me metodën e zëvendësimit

Duke zbatuar metodën e zëvendësimit të njehsohen integralet:

1. $\int (1+x)^7 dx$. 2. $\int (3-2x)^6 dx$. 3. $\int \sqrt{1-x} dx$. 4. $\int \sqrt[4]{1+x} dx$.

Zgjidhja.

$$1. \int (1+x)^7 dx = \left. \begin{array}{l} 1+x=t \\ dx=dt \end{array} \right| = \int t^7 dt = \frac{t^8}{8} + C = \frac{(1+x)^8}{8} + C.$$

$$2. \int (3-2x)^6 dx = \left. \begin{array}{l} 3-2x=t \\ -2dx=dt \\ dx=-\frac{1}{2}dt \end{array} \right| = \int t^6 \left(-\frac{1}{2}\right) dt = -\frac{1}{2} \int t^6 dt$$
$$= -\frac{1}{2} \frac{t^7}{7} + C = -\frac{t^7}{14} + C = -\frac{(3-2x)^7}{14} + C$$

3. Mënyra e parë:

$$\int \sqrt{1-x} dx = \left. \begin{array}{l} 1-x=t \\ -dx=dt \\ dx=-dt \end{array} \right| = \int \sqrt{t}(-dt) = -\int t^{\frac{1}{2}} dt = -\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$
$$= -\frac{2}{3} t^{\frac{3}{2}} + C = -\frac{2}{3} \sqrt{(1-x)^3} + C.$$

Mënyra e dytë:

$$\int \sqrt{1-x} dx = \left. \begin{array}{l} 1-x=t^2 \\ t=\sqrt{1-x} \\ -dx=2tdt \\ dx=-2tdt \end{array} \right| = \int t(-2t)dt = -2 \int t^2 dt = -\frac{2}{3} t^3 + C$$
$$= -\frac{2}{3} \sqrt{(1-x)^3} + C.$$

$$4. \int \sqrt[4]{1+x} dx = \left. \begin{array}{l} 1+x=t^4 \Rightarrow t=\sqrt[4]{1+x} \\ dx=4t^3 dt \end{array} \right| = \int t \cdot 4t^3 dt = 4 \int t^4 dt$$

$$= 4 \cdot \frac{t^5}{5} + C = \frac{4}{5} \cdot \sqrt[4]{(1+x)^5} + C.$$

Detyra për ushtrime

Të njehsohen integralet:

1. $\int (1+2x)^5 dx.$	2. $\int \left(\frac{2}{3} - x\right)^4 dx.$	3. $\int \sqrt{1 + \frac{x}{2}} dx.$
4. $\int \sqrt[3]{1-3x} dx.$	5. $\int \sqrt[5]{5x-1} dx.$	6. $\int (\sqrt{x+1} - \sqrt{x-1}) dx.$
7. $\int ((1+x)^3 - \sqrt[3]{1+x}) dx.$		

Të njehsohen integralet:

5. $\int \frac{2x+1}{x^2+x+3} dx.$	6. $\int \frac{x^3}{x^4+1} dx.$	7. $\int x \cdot \sqrt{1+2x^2} dx.$
8. $\int x^2 \cdot \sqrt{1+x^3} dx.$	9. $\int \frac{x^2-3}{\sqrt[3]{x+2}} dx.$	

Zgjidhja.

$$5. \int \frac{2x+1}{x^2+x+3} dx = \left| \begin{array}{l} x^2+x+3=t \\ (2x+1)dx=dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| = \ln(x^2+x+3) + C.$$

$$6. \int \frac{x^3}{x^4+1} dx = \left| \begin{array}{l} x^4+1=t \\ 4x^3 dx=dt \\ x^3 dx = \frac{1}{4} dt \end{array} \right| = \int \frac{\frac{1}{4} dt}{t} = \frac{1}{4} \int \frac{dt}{t} = \frac{1}{4} \ln|t| = \frac{\ln(x^4+1)}{4} + C.$$

$$7. \int x \cdot \sqrt{1+2x^2} dx = \left| \begin{array}{l} 1+2x^2=t^2 \Rightarrow t = \sqrt{1+2x^2} \\ 4xdx = 2tdt \\ xdx = \frac{1}{2} tdt \end{array} \right| = \int t \cdot \frac{1}{2} t dt$$

$$= \frac{1}{2} \cdot \frac{t^3}{3} = \frac{1}{6} \sqrt{(1+2x^2)^3} + C.$$

$$8. \int \sqrt{1+x^3} \cdot x^2 dx = \left| \begin{array}{l} 1+x^3 = t^2 \Rightarrow t = \sqrt{1+x^3} \\ 3x^2 dx = 2t dt \\ x^2 dx = \frac{2}{3} t dt \end{array} \right| = \int t \cdot \frac{2}{3} t dt = \frac{2}{3} \cdot \frac{t^3}{3} + C$$

$$= \frac{2}{9} \sqrt{(1+x^3)^3} + C$$

$$9. \int \frac{x^2-3}{\sqrt[3]{x+2}} dx = \left| \begin{array}{l} x+2 = t^3 \Rightarrow x = t^3 - 2; t = \sqrt[3]{x+2} \\ dx = 3t^2 dt \end{array} \right|$$

$$= \int \frac{(t^3-2)^2-3}{t} \cdot 3t^2 dt = 3 \int (t^6 - 4t^3 + 4 - 3) \cdot t dt$$

$$= 3 \int (t^7 - 4t^4 + t) dt = 3 \cdot \frac{t^8}{8} - 12 \cdot \frac{t^5}{5} + 3 \cdot \frac{t^2}{2}$$

$$= \frac{3}{8} \sqrt[3]{(x+2)^8} - \frac{12}{5} \sqrt[3]{(x+2)^5} + \frac{3}{2} \sqrt[3]{(x+2)^2} + C.$$

Detyra plotësuese

Të njehsohen integralet

$$8. \int \frac{2x-1}{x^2-x+4} dx.$$

$$9. \int \frac{2x+3}{x^2+3x+1} dx.$$

$$10. \int \frac{x^5}{x^6-1} dx.$$

$$11. \int \frac{2x-a}{x^2-ax+1} dx, a-\text{const.}$$

$$12. \int \frac{2x^3}{4x^4+1} dx.$$

$$13. \int \frac{x-2}{\sqrt{x-3}} dx.$$

$$14. \int x \cdot \sqrt{1-x^2} dx.$$

$$15. \int x \cdot \sqrt{2+3x^2} dx.$$

$$16. \int \frac{x(x^2-1)}{\sqrt[3]{x^2+1}} dx.$$

$$17. \int x^2 \cdot \sqrt{2+5x^3} dx.$$

$$18. \int \left(\frac{x-3}{\sqrt{x+5}} - \frac{x^2}{\sqrt{x^2+2}} \right) dx.$$

Të njehsohen integralet:

$$10. \int \frac{dx}{\sqrt{x+1} - \sqrt{x-1}}.$$

$$11. \int \frac{1+x}{\sqrt{1-x^2}} dx.$$

$$12. \int \sqrt{\frac{1-x}{1+x}} dx.$$

$$13. \int \frac{dx}{x\sqrt{x^2-1}}.$$

Zgjidhja.

10. Pas racionalizimit të emëruesit kemi:

$$\begin{aligned} I &= \int \frac{dx}{\sqrt{x+1} - \sqrt{x-1}} = \int \frac{1}{\sqrt{x+1} - \sqrt{x-1}} \cdot \frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx \\ &= \int \frac{\sqrt{x+1} + \sqrt{x-1}}{x+1 - (x-1)} dx = \int \frac{\sqrt{x+1} + \sqrt{x-1}}{2} dx \\ &= \frac{1}{2} \left(\int \sqrt{x+1} dx + \int \sqrt{x-1} dx \right) = \frac{1}{2} (I_1 + I_2), \end{aligned}$$

$$\text{ku } I_1 = \int \sqrt{x+1} dx; I_2 = \int \sqrt{x-1} dx.$$

Zgjidhim veçmas integralet I_1, I_2 . Merret:

$$I_1 = \int \sqrt{x+1} dx = \left. \begin{array}{l} x+1 = t^2 \\ t = \sqrt{x+1} \\ dx = 2t dt \end{array} \right| = \int t \cdot 2t dt = 2 \cdot \frac{t^3}{3} = \frac{2}{3} \sqrt{(x+1)^3} + C.$$

$$I_2 = \int \sqrt{x-1} dx = \left. \begin{array}{l} x-1 = t^2 \\ t = \sqrt{x-1} \\ dx = 2t dt \end{array} \right| = \int t \cdot 2t dt = 2 \cdot \frac{t^3}{3} = \frac{2}{3} \sqrt{(x-1)^3} + C.$$

Pra

$$I = \frac{1}{2} \left(\frac{2}{3} \sqrt{(x+1)^3} + \frac{2}{3} \sqrt{(x-1)^3} \right) + C = \frac{1}{3} \left(\sqrt{(x+1)^3} + \sqrt{(x-1)^3} \right) + C.$$

$$11. I = \int \frac{1+x}{\sqrt{1-x^2}} dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{xdx}{\sqrt{1-x^2}} = \arcsin x + I_1.$$

$$I_1 = \int \frac{xdx}{\sqrt{1-x^2}} = \left. \begin{array}{l} 1-x^2 = t^2 \\ -2xdx = 2t dt \\ xdx = -t dt \end{array} \right| = \int \frac{-t dt}{t} = -\int dt = -t = -\sqrt{1-x^2}$$

$$\text{Pra, } I = \arcsin x - \sqrt{1-x^2} + C$$

12. Pasi të kryejmë racionalizimin merret

$$\int \sqrt{\frac{1-x}{1+x}} dx = \int \sqrt{\frac{1-x}{1+x}} \cdot \frac{\sqrt{1-x}}{\sqrt{1-x}} dx = \int \frac{1-x}{\sqrt{1-x^2}} dx.$$

Integrali që morëm është i ngjashëm me integralin e detyrës paraprake.

$$\text{Pas zgjidhjes merret } \int \sqrt{\frac{1-x}{1+x}} dx = \arcsin x - \sqrt{1-x^2} + C.$$

$$\begin{aligned}
 13. \int \frac{dx}{x\sqrt{x^2-1}} &= \int \frac{dx}{x\sqrt{x^2\left(1-\frac{1}{x^2}\right)}} = \int \frac{dx}{x|x|\sqrt{1-\left(\frac{1}{x}\right)^2}} \\
 &= \int \frac{dx}{x \cdot x \operatorname{sgn} x \sqrt{1-\left(\frac{1}{x}\right)^2}} = \frac{1}{\operatorname{sgn} x} \cdot \int \frac{dx}{x^2 \sqrt{1-\left(\frac{1}{x}\right)^2}} \\
 &= \operatorname{sgn} x \cdot \int \frac{dx}{x^2 \sqrt{1-\left(\frac{1}{x}\right)^2}} = \left. \begin{array}{l} \frac{1}{x} = t \\ -\frac{1}{x^2} = dt \end{array} \right| = \operatorname{sgn} x \int \frac{-dt}{\sqrt{1-t^2}} \\
 &= -\operatorname{sgn} x \cdot \arcsin t + C = -\operatorname{sgn} x \cdot \arcsin \frac{1}{x} + C.
 \end{aligned}$$

Detyra për ushtrime

Të njehsohen integralet:

19. $\int \frac{dx}{\sqrt{x-1} + \sqrt{x+1}}$.	20. $\int \frac{dx}{\sqrt{2x-1} - \sqrt{2x+1}}$.	21. $\int \frac{3 + \frac{x}{2}}{\sqrt{1-x^2}} dx$.
22. $\int \frac{1 - \frac{x}{3}}{\sqrt{1-(2x)^2}} dx$.	23. $\int \frac{x-1}{\sqrt{1-9x^2}} dx$.	24. $\int \frac{1-x}{\sqrt{1-4x^2}} dx$.
25. $\int \sqrt{\frac{1+x}{1-x}} dx$.	26. $\int \frac{xdx}{\sqrt{x^4-x^2}}$.	27. $\int \frac{dx}{x\sqrt{x^2+1}}$.

Të njehsohen integralet:

14. $\int \frac{dx}{x(1+2\ln x)}$.	15. $\int \frac{dx}{x\sqrt{1-\ln^2 x}}$.	16. $\int \frac{dx}{x \ln x \ln(\ln x)}$.
17. $\int \frac{dx}{x \cos^2 \ln x}$.	18. $\int \frac{1}{1-x^2} \ln \frac{1+x}{1-x} dx$.	
19. $\int \sqrt{\frac{\ln(x + \sqrt{1+x^2})}{1+x^2}} dx$.		

Zgjidhja.

$$14. \int \frac{dx}{x(1+2\ln x)} = \left| \frac{(1+2\ln x) = t}{\frac{dx}{x} = \frac{1}{2} dt} \right| = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln |t|$$

$$= \frac{1}{2} \ln |1+2\ln x| + C.$$

$$15. \int \frac{dx}{x\sqrt{1-\ln^2 x}} = \left| \frac{\ln x = t}{\frac{dx}{x} = dt} \right| = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin(\ln x) + C.$$

$$16. \int \frac{dx}{x \ln x \ln(\ln x)} = \left| \frac{\ln(\ln x) = u}{\frac{dx}{x \ln x} = du} \right| = \int \frac{du}{u} = \ln |u| + C = \ln |\ln(\ln x)| + C.$$

$$17. \int \frac{dx}{x \cos^2 \ln x} = \left| \frac{\ln x = t}{\frac{dx}{x} = dt} \right| = \int \frac{dt}{\cos^2 t} = \tan t + C = \tan(\ln x) + C.$$

$$18. I = \int \frac{1}{1-x^2} \ln \frac{1+x}{1-x} dx$$

$$\text{Zëvendësojmë } \ln \frac{1+x}{1-x} = u$$

Pas diferencimit merret:

$$\frac{1}{1+x} \cdot \left(\frac{1+x}{1-x} \right)' dx = du$$

$$\frac{1-x}{1+x} \cdot \frac{(1+x)'(1-x) - (1+x)(1-x)'}{(1-x)^2} dx = du$$

$$\frac{1}{1+x} \cdot \frac{1-x - (1-x)(-1)}{1-x} dx = du$$

$$\frac{1-x+1-x}{1-x^2} dx = du$$

$$\frac{1}{1-x^2} dx = \frac{1}{2} du$$

Pra, kemi:

$$I = \int u \frac{du}{2} = \frac{u^2}{4} = \frac{\left(\ln \frac{1+x}{1-x} \right)^2}{4} = \frac{1}{4} \ln^2 \frac{1+x}{1-x} + C.$$

$$19. I = \int \sqrt{\frac{\ln(x + \sqrt{1+x^2})}{1+x^2}} dx = \int \frac{\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx$$

Zëvendësojmë $\ln(x + \sqrt{1+x^2}) = u$

Pas diferencimit merret:

$$\frac{(x + \sqrt{1+x^2})'}{x + \sqrt{1+x^2}} dx = du$$

$$\frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} dx = du$$

$$\frac{x + \sqrt{1+x^2}}{x + \sqrt{1+x^2}} dx = du$$

Pra $\frac{dx}{\sqrt{1+x^2}} = du.$

Merret:

$$I = \int \sqrt{u} du = \int u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3} \sqrt{\ln^3(x + \sqrt{1+x^2})} + C.$$

Detyra për ushtrime

Të njehsohen integralet:

$$28. \int \frac{dx}{2x(\ln x + 1)}.$$

$$29. \int \frac{dx}{x\sqrt{1-\ln x}}.$$

$$30. \int \frac{\sin x}{\ln(\cos x)} dx.$$

$$31. \int \frac{\cos x}{\ln(\sin x)} dx.$$

$$32. \int \frac{dx}{\cos^2 x \ln(\operatorname{ctg} x)}.$$

$$33. \int \frac{dx}{x \sin^2 \ln x}.$$

$$34. \int \frac{1}{1-x^2} \ln \frac{1-x}{1+x} dx.$$

$$35. \int \sqrt{\frac{\ln(x - \sqrt{1+x^2})}{1+x^2}} dx.$$

Të njehsohen integralet:

$$20. \int e^{\cos x} \sin x dx.$$

$$21. \int \frac{e^x + 1}{e^x + x} dx.$$

$$22. \int \frac{e^x}{e^x + \frac{1}{2}} dx.$$

$$23. \int x^2 e^{-x^3} dx.$$

$$24. \int \frac{e^{\frac{1}{x}}}{x^2} dx.$$

$$25. \int \frac{dx}{e^x + e^{-x}}.$$

$$26. \int \frac{e^x}{\sqrt{e^x + 1}} dx.$$

Zgjidhja.

$$20. \int e^{\cos x} \sin x dx = \left| \begin{array}{l} \cos x = t \\ -\sin x dx = dt \end{array} \right| = -\int e^t dt = -e^t = -e^{\cos x} + C.$$

$$21. \int \frac{e^x + 1}{e^x + x} dx = \left| \begin{array}{l} e^x + x = t \\ (e^x + 1) dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln |t| = \ln |e^x + x| + C.$$

$$22. \int \frac{e^x}{e^x + \frac{1}{2}} dx = \left| \begin{array}{l} e^x + \frac{1}{2} = t \\ e^x dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln |t| = \ln \left(e^x + \frac{1}{2} \right) + C.$$

$$23. \int x^2 e^{-x^3} dx = \left| \begin{array}{l} -x^3 = t \\ -3x^2 dx = dt \\ x^2 dx = -\frac{1}{3} dt \end{array} \right| = -\frac{1}{3} \int e^t dt = -\frac{1}{3} e^t = -\frac{1}{3} e^{-x^3} + C.$$

$$24. \int \frac{e^{\frac{1}{x}}}{x^2} dx = \left| \begin{array}{l} \frac{1}{x} = t \\ -\frac{1}{x^2} dx = -dt \end{array} \right| = -\int e^t dt = -e^t = -e^{\frac{1}{x}} + C.$$

$$25. \int \frac{dx}{e^x + e^{-x}} = \int \frac{dx}{e^x + \frac{1}{e^x}} = \int \frac{dx}{\frac{e^{2x} + 1}{e^x}} = \int \frac{e^x dx}{e^{2x} + 1}$$

$$= \int \frac{e^x dx}{(e^x)^2 + 1} = \left| \begin{array}{l} e^x = t \\ e^x dx = dt \end{array} \right| = \int \frac{dt}{t^2 + 1} = \arctan t = \arctan e^x + C.$$

$$26. \int \frac{e^x}{\sqrt{e^x + 1}} dx = \left| \begin{array}{l} e^x + 1 = t^2 \Rightarrow t = \sqrt{e^x + 1} \\ e^x dx = 2t dt \end{array} \right| = \int \frac{2t dt}{t} = 2\sqrt{e^x + 1} + C.$$

Detyra për ushtrime

Të njehsohen integralet:

$$36. \int e^{\sin x} \cos x dx.$$

$$37. \int \frac{e^{\operatorname{tg} x}}{\cos^2 x} dx.$$

$$38. \int \frac{dx}{e^{-x} + 1}.$$

39. $\int \frac{2e^{2x} + 1}{e^{2x} + x} dx.$

40. $\int x^2 e^{-2x^3} dx.$

41. $\int x e^{x^2+a^2} dx.$

42. $\int \frac{e^x dx}{\sqrt[3]{e^x + 3}}.$

Të njehsohen integralet:

27. $\int \frac{dx}{x^2 + a^2}.$

28. $\int \frac{x^2 dx}{9 + x^6}.$

29. $\int \frac{8x - 3}{x^2 + 7} dx.$

Zgjidhja.

$$27. \int \frac{dx}{x^2 + a^2} = \int \frac{\frac{dx}{a^2}}{\frac{x^2 + a^2}{a^2}} = \frac{1}{a^2} \int \frac{dx}{\left(\frac{x}{a}\right)^2 + 1} \left| \begin{array}{l} \frac{x}{a} = t \\ dx = a dt \end{array} \right| = \frac{1}{a^2} \int \frac{a dt}{t^2 + 1}$$

$$= \frac{1}{a} \int \frac{dt}{t^2 + 1} = \frac{1}{a} \arctan t = \frac{1}{a} \arctan \frac{x}{a} + C.$$

$$28. I = \int \frac{x^2 dx}{9 + x^6} = \int \frac{x^2 dx}{3^2 + (x^3)^2} = \left| \begin{array}{l} x^3 = t \\ 3x^2 dx = dt \\ x^2 dx = \frac{dt}{3} \end{array} \right| = \int \frac{\frac{dt}{3}}{3^2 + t^2} = \frac{1}{3} \int \frac{dt}{3^2 + t^2}.$$

Në bazë të detyrës paraprake

$$\int \frac{dt}{3^2 + t^2} = \frac{1}{3} \arctan \frac{t}{3} + C_1 = \frac{1}{3} \arctan \frac{x^3}{3} + C_1.$$

$$\text{Pra } I = \frac{1}{9} \arctan \frac{x^3}{3} + C.$$

$$29. \int \frac{8x - 3}{x^2 + 7} dx = 8 \int \frac{8x}{x^2 + 7} dx = -3 \int \frac{dx}{x^2 + (\sqrt{7})^2} = 8I_1 - 3I_2, \text{ ku}$$

$$I_1 = \int \frac{xdx}{x^2 + 7}; \quad I_2 = \int \frac{dx}{x^2 + (\sqrt{7})^2}$$

$$I_1 = \int \frac{xdx}{x^2 + 7} = \left| \begin{array}{l} x^2 + 7 = t \\ 2xdx = dt \\ xdx = \frac{dt}{2} \end{array} \right| = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln |t| = \frac{1}{2} \ln(x^2 + 7).$$

Pra

$$\begin{aligned}
 I &= 8 \cdot \frac{1}{2} \ln(x^2 + 7) - 3 \frac{1}{\sqrt{7}} \arctan \frac{x}{\sqrt{7}} + C \\
 &= 4 \ln(x^2 + 7) - \frac{3}{\sqrt{7}} \arctan \frac{x}{\sqrt{7}} + C.
 \end{aligned}$$

Detyra për ushtrime

Të njehsohen integralet

43. $\int \frac{dx}{2x^2 + a^2}.$	44. $\int \frac{dx}{2x^2 + 3}.$	45. $\int \frac{x^3}{7 + x^8} dx.$
46. $\int \frac{x^2}{2x^6 + 5} dx.$	47. $\int \frac{9x - 1}{x^2 + 15} dx.$	48. $\int \frac{dx}{(ex)^2 + \pi^2}.$

Të njehsohen integralet:

30. $\int \frac{a^{\arctan x}}{1 + x^2} dx, a \in \mathbb{R} \setminus \{1\}, a > 0.$	31. $\int \frac{dx}{\arcsin x \sqrt{1 - x^2}}.$
32. $\int \frac{\arctan x + x \ln(1 + x^2)}{1 + x^2} dx.$	33. $\int \frac{e^{\arctg 2x} + x}{1 + 4x^2} dx.$

Zgjidhja.

$$\mathbf{30.} \int \frac{a^{\arctan x}}{1 + x^2} dx = \left. \frac{\arctan x = u}{\frac{dx}{1 + x^2} = du} \right| = \int a^u du = \frac{a^u}{\ln |a|} = \frac{a^{\arctan x}}{\ln a} + C.$$

$$\mathbf{31.} \int \frac{dx}{\arcsin x \sqrt{1 - x^2}} = \left. \frac{\arcsin x = u}{\frac{dx}{\sqrt{1 - x^2}} = du} \right| = \int \frac{du}{u} = \ln |u| = \ln |\arcsin x| + C.$$

$$\begin{aligned}
 \mathbf{32.} \quad I &= \int \frac{\arctan x + x \ln(1 + x^2)}{1 + x^2} dx = \int \frac{\arctan x}{1 + x^2} dx + \int \frac{x \ln(1 + x^2)}{1 + x^2} dx \\
 &= I_1 + I_2,
 \end{aligned}$$

$$\text{ku } I_1 = \int \frac{\arctan x}{1 + x^2} dx = \left. \frac{\arctan x = u}{\frac{dx}{1 + x^2} = du} \right| = \int u du = \frac{u^2}{2} = \frac{(\arctan x)^2}{2}.$$

$$I_2 = \int \frac{\ln(1 + x^2)}{1 + x^2} dx = \left. \frac{\ln(1 + x^2) = u}{\frac{1}{1 + x^2} x dx = \frac{du}{2}} \right| = \frac{1}{2} \int u du = \frac{u^2}{4} = \frac{\ln^2(1 + x^2)}{4}.$$

Pra

$$I = \frac{(\arctan x)^2}{2} + \frac{\ln^2(1+x^2)}{4} + C.$$

$$33. I = \int \frac{e^{\arctan 2x} + x}{1+4x^2} dx = \int \frac{e^{\arctan 2x}}{1+4x^2} dx + \int \frac{xdx}{1+4x^2} = I_1 + I_2,$$

$$I_1 = \int \frac{e^{\arctan 2x}}{1+4x^2} dx = \left. \begin{array}{l} \arctan 2x = u \\ \frac{dx}{1+4x^2} = \frac{1}{2} du \end{array} \right| = \frac{1}{2} \int e^u du = \frac{e^u}{2} = \frac{e^{\arctan 2x}}{2}.$$

$$I_2 = \int \frac{xdx}{1+4x^2} = \left. \begin{array}{l} 1+4x^2 = t \\ 8xdx = dt \end{array} \right| = \frac{1}{8} \int \frac{dt}{t} = \frac{1}{8} \ln |t| = \frac{1}{8} \ln(1+4x^2).$$

$$\text{Pra, } I = \frac{e^{\arctan 2x}}{2} + \frac{1}{8} \ln(1+4x^2) + C.$$

Detyra për ushtrime

Të njehsohen integralet:

$$49. \int \frac{3^{\arctan x}}{1+x^2} dx.$$

$$50. \int \frac{\pi^{\arctan 3x}}{1+9x^2} dx.$$

$$51. \int \frac{\sqrt[3]{\arcsin^2 x}}{\sqrt{1-x^2}} dx.$$

$$52. \int \frac{x \ln(1+3x^2)}{1+3x^2} dx.$$

$$53. \int \frac{e^{\arctan 4x} + e^{\arctan 4x}}{x^2 + \frac{1}{16}} dx.$$

Të njehsohen integralet:

$$34. \int \sin ax dx.$$

$$35. \int \cos ax dx.$$

$$36. \int \tan ax dx.$$

$$37. \int \cot ax dx.$$

$$38. \int \frac{\cos 2x}{\sin x \cos x} dx.$$

$$39. \int \sin^4 x dx.$$

$$40. \int \tan^4 x dx.$$

$$41. \int \sin^5 x dx.$$

$$42. \int \cot^5 x dx.$$

$$43. \int \sin x \sin 2x dx.$$

$$44. \int \cos x \cos 2x \cos 3x dx.$$

$$45. \int \frac{dx}{2 \sin^2 x + \cos^2 x}.$$

$$46. \int \frac{\sin 2x}{\tan^4 x} dx.$$

$$47. \int \frac{\sin^3 x}{\cos^2 x} dx.$$

$$48. \int \frac{\sin x - \sin^3 x}{2 \cos^2 x + \sin^2 x} dx.$$

$$49. \int \frac{dx}{\sin x}.$$

$$50. \int \sin^4 x \cdot \cos^4 x dx.$$

Zgjidhja.

$$34. \int \sin ax \, dx = \left| \begin{array}{l} ax = t \\ adx = dt \\ dx = \frac{1}{a} dt \end{array} \right| = \frac{1}{a} \int \sin t \, dt = -\frac{1}{a} \cos t = -\frac{1}{a} \cos ax + C.$$

$$35. \int \cos ax \, dx = \left| \begin{array}{l} ax = t \\ adx = dt \\ dx = \frac{1}{a} dt \end{array} \right| = \frac{1}{a} \int \cos t \, dt = \frac{1}{a} \sin t = \frac{1}{a} \sin ax + C.$$

$$36. \int \tan ax \, dx = \int \frac{\sin ax}{\cos ax} \, dx = \left| \begin{array}{l} \cos ax = u \\ \sin ax \, dx = -\frac{1}{a} du \end{array} \right| \\ = -\frac{1}{a} \int \frac{du}{u} = -\frac{1}{a} \ln |u| = -\frac{1}{a} \ln |\cos ax| + C.$$

$$37. \int \cot ax \, dx = \int \frac{\cos ax}{\sin ax} \, dx = \left| \begin{array}{l} \sin ax = u \\ \cos ax \, dx = \frac{1}{a} du \end{array} \right| \\ = \frac{1}{a} \int \frac{du}{u} = \frac{1}{a} \ln |u| = \frac{1}{a} \ln |\sin ax| + C.$$

38. Mënyra e parë:

$$I = \int \frac{\cos 2x}{\sin x \cos x} \, dx = \int \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} \, dx = \int \frac{\cos x}{\sin x} \, dx - \int \frac{\sin x}{\cos x} \, dx \\ = \int \cot x \, dx - \int \tan x \, dx = \ln |\sin x| + \ln |\cos x| \\ = \ln |\sin x \cdot \cos x| + C.$$

Mënyra e dytë:

$$I = \int \frac{\cos 2x}{\sin x \cos x} \, dx = \int \frac{2 \cos 2x}{2 \sin x \cos x} \, dx = 2 \int \frac{\cos 2x}{\sin 2x} \, dx = 2 \int \cot 2x \, dx \\ = \ln |\sin 2x| + C_1.$$

Shënim.

Lexuesi mund të ketë përshtypjen se rezultatet që marrëm nga zgjidhja në dy mënyrat janë të ndryshme por lehtë vërejmë se:

$$\begin{aligned}\ln |\sin 2x| + C_1 &= \ln |2 \sin x \cos x| + C_1 = \ln 2 + \ln |\sin x \cos x| + C_1 \\ &= \ln |\sin x \cos x| + C.\end{aligned}$$

$$\begin{aligned}39. \int \sin^4 x dx &= \int (\sin^2 x)^2 dx = \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) dx \\ &= \frac{1}{4} \int dx - \frac{1}{2} \int \cos 2x dx + \frac{1}{4} \int \frac{1 + \cos 4x}{2} dx \\ &= \frac{1}{4} x - \frac{1}{2} I_1 + \frac{1}{4} I_2.\end{aligned}$$

$$I_1 = \int \cos 2x dx = \left. \begin{array}{l} 2x = u \\ dx = \frac{1}{2} du \end{array} \right| = \frac{1}{2} \int \cos u du = \frac{1}{2} \sin u = \frac{1}{2} \sin 2x + C_1.$$

$$I_2 = \int \frac{1 + \cos 4x}{2} dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos 4x dx = \frac{1}{2} x + \frac{1}{8} \sin 4x + C_2.$$

Përfundimisht merret:

$$I = \frac{1}{4} x - \frac{1}{2} \left(\frac{1}{2} \sin 2x \right) + \frac{1}{4} \left(\frac{1}{2} x + \frac{1}{8} \sin 4x \right) = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$

$$\begin{aligned}40. \int \tan^4 x dx &= \int \tan^2 \tan^2 x dx = \int \tan^2 x \left(\frac{1}{\cos^2 x} - 1 \right) dx \\ &= \int \tan^2 x \frac{1}{\cos^2 x} dx - \int \tan^2 x dx = I_1 - I_2.\end{aligned}$$

$$I_1 = \int \tan^2 x \frac{1}{\cos^2 x} dx = \left. \begin{array}{l} \tan x = u \\ \frac{dx}{\cos^2 x} = du \end{array} \right| = \int u^2 du = \frac{u^3}{3} = \frac{\tan^3 x}{3}.$$

$$\begin{aligned}I_2 &= \int \tan^2 x dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1 - \cos^2 x}{\cos^2 x} dx = \int \frac{dx}{\cos^2 x} - \int dx \\ &= \tan x - x + C.\end{aligned}$$

Përfundojmë se

$$\int \tan^4 x dx = \frac{\tan^3 x}{3} - \tan x + x + C.$$

$$41. \int \sin^5 x dx = \int \sin x \sin^4 x dx = \int \sin x (1 - \cos^2 x)^2 dx = \left. \begin{array}{l} \cos x = t \\ \sin x dx = -dt \end{array} \right|$$

$$\begin{aligned}
 &= -\int (1-t^2)^2 dt = -\int dt + 2\int t^2 dt - \int t^4 dt = -t + 2\frac{t^3}{3} - \frac{t^5}{5} \\
 &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.
 \end{aligned}$$

$$\begin{aligned}
 42. \int \cot^5 x dx &= \int \frac{\cos^5 x}{\sin^5 x} dx = \int \frac{\cos x \cdot \cos^4 x}{\sin^5 x} dx = \int \frac{\cos x (1 - \sin^2 x)^2}{\sin^5 x} dx \\
 &= \left| \begin{array}{l} \sin x = u \\ \cos x dx = du \end{array} \right| = \int \frac{(1-u^2)^2}{u^5} du = \int \frac{1-2u^2+u^4}{u^5} du \\
 &= \int u^{-5} du - 2\int u^{-3} du + \int \frac{du}{u} = -\frac{1}{4u^4} + \frac{1}{u^2} + \ln |u| \\
 &= -\frac{1}{4\sin^4 x} + \frac{1}{\sin^2 x} + \ln |\sin x| + C.
 \end{aligned}$$

Në detyrat vijuese zbatohen formulat:

$$\sin x \cdot \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\cos x \cdot \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$

$$\sin x \cdot \cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y)).$$

$$\begin{aligned}
 43. \int \sin x \sin 2x dx &= \frac{1}{2} \int (\cos(x-2x) - \cos(x+2x)) dx \\
 &= \frac{1}{2} \left(\int \cos x dx - \int \cos 3x dx \right) = \frac{1}{2} \sin x - \frac{1}{6} \sin 3x + C.
 \end{aligned}$$

$$\begin{aligned}
 44. \int \cos x \cos 2x \cos 3x dx &= \int \cos 2x \cdot \frac{1}{2}(\cos(x-3x) + \cos(x+3x)) dx \\
 &= \frac{1}{2} \int \cos 2x (\cos(-2x) + \cos 4x) dx \\
 &= \frac{1}{2} \left(\int \cos 2x \cos 2x dx + \int \cos 2x \cos 4x dx \right) \\
 &= \frac{1}{4} \left(\int dx + \int \cos 4x dx + \int \cos 2x dx + \int \cos 6x dx \right) \\
 &= \frac{1}{4} \left(x + \frac{1}{2} \sin 2x + \frac{1}{4} \sin 4x + \frac{1}{6} \sin 6x \right) + C.
 \end{aligned}$$

$$45. \int \frac{dx}{2\sin^2 x + \cos^2 x} = \int \frac{\frac{dx}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + 1} = \int \frac{\frac{dx}{\cos^2 x}}{2\tan^2 x + 1} = \left| \begin{array}{l} \tan x = u \\ \frac{dx}{\cos^2 x} = du \end{array} \right|$$

$$\begin{aligned}
&= \int \frac{du}{2u^2 + 1} = \int \frac{du}{(\sqrt{2}u)^2 + 1} = \sqrt{2} \arctan \sqrt{2}u \\
&= \sqrt{2} \arctan(\sqrt{2} \tan x) + C.
\end{aligned}$$

$$\begin{aligned}
46. \int \frac{\sin 2x}{\tan^4 x} dx &= \int \frac{2 \sin x \cos x}{\frac{\sin^4 x}{\cos^4 x}} dx = 2 \int \frac{\sin x \cos x \cos^4 x}{\sin^4 x} dx \\
&= 2 \int \frac{\sin x (1 - \sin^2 x)^2 \cos x}{\sin^4 x} dx = \left| \begin{array}{l} \sin x = u \\ \cos x dx = du \end{array} \right| \\
&= 2 \int \frac{u(1-u^2)^2}{u^4} du = 2 \int \frac{du}{u^3} - 4 \int \frac{du}{u^2} + 2 \int \frac{du}{u} \\
&= -\frac{1}{u^2} + \frac{4}{u} + 2 \ln |u| = -\frac{1}{\sin^2 x} + \frac{4}{\sin x} + 2 \ln |\sin x| + C.
\end{aligned}$$

$$\begin{aligned}
47. \int \frac{\sin^3 x}{\cos^2 x} dx &= \int \frac{\sin x \sin^2 x}{\cos^2 x} dx = \int \frac{\sin x (1 - \cos^2 x)}{\cos^2 x} dx \\
&= \left| \begin{array}{l} \cos x = u \\ \sin x dx = -du \end{array} \right| = -\int \frac{1-u^2}{u^2} du = \frac{1}{u} + u = \frac{1}{\cos x} + \cos x + C.
\end{aligned}$$

$$\begin{aligned}
48. \int \frac{\sin x - \sin^3 x}{2 \cos^2 x + \sin^2 x} dx &= \int \frac{\sin x (1 - \sin^2 x)}{\cos^2 x + 1} dx = \int \frac{\sin x \cos^2 x}{\cos^2 x + 1} dx \\
&= \left| \begin{array}{l} \cos x = t \\ \sin x dx = -dt \end{array} \right| = -\int \frac{t^2}{t^2 + 1} dt = -\int dt + \int \frac{dt}{t^2 + 1} \\
&= -t + \arctan t = -\cos x + \arctan(\cos x) + C.
\end{aligned}$$

$$\begin{aligned}
49. \int \frac{dx}{\sin x} &= \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \int \frac{\frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} dx \\
&= \frac{1}{2} \left(\int \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} dx + \int \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} dx \right) = \frac{1}{2} (I_2 + I_2).
\end{aligned}$$

$$I_1 = \int \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} dx = \left. \begin{array}{l} \cos \frac{x}{2} = u \\ -\frac{1}{2} \sin \frac{x}{2} dx = du \\ \sin \frac{x}{2} dx = -2 du \end{array} \right| = \int \frac{du}{u} = -2 \ln |u| = -2 \ln \left| \cos \frac{x}{2} \right|.$$

$$I_2 = \int \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} dx = \left. \begin{array}{l} \sin \frac{x}{2} = u \\ \cos \frac{x}{2} dx = 2du \end{array} \right| = 2 \int \frac{du}{u} = 2 \ln |u| = 2 \ln \left| \sin \frac{x}{2} \right|.$$

$$\text{Pra } I = \frac{1}{2} \left(-2 \ln \left| \cos \frac{x}{2} \right| + 2 \ln \left| \sin \frac{x}{2} \right| \right) = \ln \left| \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right| = \ln \left| \tan \frac{x}{2} \right| + C.$$

$$\begin{aligned} 50. \int \sin^4 x \cdot \cos^4 x dx &= \int (\sin x \cos x)^4 dx = \int \left(\frac{2 \sin x \cos x}{2} \right)^4 dx \\ &= \int \frac{(\sin^2 2x)^2}{16} dx = \frac{1}{16} \int \left(\frac{1 - \cos 4x}{2} \right)^2 dx \\ &= \frac{1}{64} \int (1 - 2 \cos 4x + \cos^2 4x) dx \\ &= \frac{1}{64} \left(\int dx - 2 \int \cos 4x dx + \int \cos^2 4x dx \right) \\ &= \frac{1}{64} \left(x - 2 \cdot \frac{1}{4} \sin 4x + \int \frac{1 + \cos 8x}{2} dx \right) \\ &= \frac{1}{64} \left(x - \frac{1}{2} \sin 4x + \frac{1}{2} x + \frac{1}{16} \sin 8x \right) \\ &= \frac{1}{64} \left(\frac{3}{2} x - \frac{1}{2} \sin 4x + \frac{1}{16} \sin 8x \right) + C. \end{aligned}$$

Detyra për ushtrime

Të njehsohen integralet:

$$54. \int \frac{\cos x}{\sin \frac{x}{2} \cos \frac{x}{2}} dx.$$

$$55. \int \cos^4 x dx.$$

$$56. \int \cot^4 x dx.$$

$$57. \int \cos^5 x dx.$$

$$58. \int \tan^5 x dx.$$

$$59. \int \sin 3x \sin x dx.$$

$$\begin{array}{lll}
 60. \int \sin x \sin 2x \cos 3x dx. & 61. \int \frac{dx}{\sin^2 x + 5 \cos^2 x}. & \\
 62. \int \frac{\cos 2x}{\cot^4 x} dx. & 63. \int \frac{\sin 2x}{\cot^4 x} dx. & 64. \int \frac{\sin^5 x}{\cos^2 x} dx. \\
 65. \int \frac{\cos^2 x - \sin^2 x}{\sin x + \sin^3 x} dx. & 66. \int \frac{dx}{\cos x}. & 67. \int \frac{dx}{\sin \frac{x}{2}}.
 \end{array}$$

Të njehsohen integralet

$$\begin{array}{ll}
 51. \int \frac{dx}{\sqrt{x^2 + a^2}}. & 52. \int \frac{dx}{x\sqrt{a^2 + x^2}}. \\
 53. \int \sqrt{a^2 - x^2} dx. & 54. \int \frac{dx}{\sqrt{(1+x^2)^3}}.
 \end{array}$$

Zgjidhja.

$$\begin{array}{l}
 \left. \begin{array}{l}
 \sqrt{x^2 + a^2} = t - x, t = x + \sqrt{x^2 + a^2} \\
 x^2 + a^2 = t^2 - 2tx + x^2 \\
 2tx = t^2 - a^2 \Rightarrow x = \frac{t^2 - a^2}{2t}, dx = \frac{t^2 + a^2}{2t^2} dt \\
 \sqrt{x^2 + a^2} = t - \frac{t^2 - a^2}{2t} \\
 \sqrt{x^2 + a^2} = \frac{t^2 + a^2}{2t}
 \end{array} \right| \\
 \\
 = \int \frac{\frac{t^2 + a^2}{2t}}{\frac{t^2 + a^2}{2t}} dt = \int \frac{dt}{t} = \ln |t| = \ln |x + \sqrt{x^2 + a^2}| + C.
 \end{array}$$

$$\begin{array}{l}
 \left. \begin{array}{l}
 x = a \tan u \\
 dx = a \frac{du}{\cos^2 u} \\
 \sqrt{a^2 + x^2} = a \frac{1}{\cos u}
 \end{array} \right| = \int \frac{a \frac{du}{\cos^2 u}}{a \tan u \sqrt{a^2 + a^2 \tan^2 u}}
 \end{array}$$

$$\begin{aligned}
 &= \int \frac{\frac{du}{\cos^2 u}}{a \tan u \sqrt{1 + \tan^2 u}} = \int \frac{\frac{du}{\cos^2 u}}{a \sin u \frac{1}{\cos u}} = \frac{1}{a} \int \frac{du}{\sin u} \\
 &= \frac{1}{a} \ln \left| \tan \frac{u}{2} \right|.
 \end{aligned}$$

Meqë $\cos u = \frac{a}{\sqrt{a^2 + x^2}}$ atëherë $\sin u = \frac{x}{\sqrt{a^2 + x^2}}$ dhe

$$\tan \frac{u}{2} = \frac{\sin u}{1 + \cos u} = \frac{\frac{x}{\sqrt{a^2 + x^2}}}{1 + \frac{a}{\sqrt{a^2 + x^2}}} = \frac{x}{a + \sqrt{a^2 + x^2}}.$$

Pra $\int \frac{dx}{x\sqrt{a^2 + x^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right| + C.$

$$\begin{aligned}
 53. \int \sqrt{a^2 - x^2} dx &= \left. \begin{array}{l} x = a \sin t \\ dx = a \cos t dt \\ \sin t = \frac{x}{a} \\ t = \arcsin \frac{x}{a} \end{array} \right| = \int \sqrt{a^2 - a^2 \sin^2 t} \cdot a \cos t dt \\
 &= a^2 \int \sqrt{1 - \sin^2 t} \cos t dt = a^2 \int \cos^2 t dt = a^2 \int \frac{1 + \cos 2t}{2} dt \\
 &= \frac{a^2}{2} \left(\int dt + \int \cos 2t dt \right) = \frac{a^2}{2} \left(t + \frac{1}{2} \sin 2t \right) = \\
 &= \frac{a^2}{2} \left(t + \frac{1}{2} 2 \sin t \cos t \right) = \frac{a^2}{2} \left(\arcsin \frac{x}{a} + \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}} \right) \\
 &= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C.
 \end{aligned}$$

$$54. \int \frac{dx}{\sqrt{(1+x^2)^3}} = \left. \begin{array}{l} x = \tan t \\ dx = \frac{dt}{\cos^2 t} \end{array} \right| = \int \frac{\frac{dt}{\cos^2 t}}{\sqrt{(1 + \tan^2 t)^3}} = \int \frac{dt}{\cos^2 t \sqrt{\left(1 + \frac{\sin^2 t}{\cos^2 t}\right)^3}}$$

$$\begin{aligned}
 &= \int \frac{dt}{\cos^2 t \frac{1}{\cos^3 t}} = \int \cos t \, dt = \sin t + C = \frac{\tan t}{\sqrt{1 + \tan^2 t}} \\
 &= \frac{x}{\sqrt{1 + x^2}} + C.
 \end{aligned}$$

Shënim.

$$\sin^2 t = \frac{\sin^2 t}{\sin^2 t + \cos^2 t} = \frac{\frac{\sin^2 t}{\cos^2 t}}{\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t}} = \frac{\tan^2 t}{1 + \tan^2 t} \Rightarrow \sin t = \frac{\tan t}{\sqrt{1 + \tan^2 t}}.$$

Detyra për ushtrime

Të njehsohen integralet:

68. $\int \frac{dx}{\sqrt{x^2 - a^2}}$

69. $\int \frac{dx}{x\sqrt{x^2 + 1}}$

70. $\int \frac{dx}{\sqrt{9 - x^2}}$

71. $\int \frac{dx}{\sqrt{(a^2 + x^2)^3}}$