3. Integrimi me metodën parciale

Të njehsohen integralet:

1. $\int \ln x\, dx.$ **2.** $\int x \ln x\, dx.$ **3.** $\int x^n \ln x\, dx.$ **4.** $\int x^4 \ln^3 x\, dx.$

Zgjidhja.

1. $$\int \ln x\, dx = \left| \begin{array}{l} \ln x = u \\ \dfrac{dx}{x} = du \\ \int dx = v \Rightarrow v = x \end{array} \right| = x\ln x - \int x \cdot \frac{dx}{x} = x \ln x - \int dx$$

$$= x\ln x - x + C.$$

2. $$\int x \ln x\, dx = \left| \begin{array}{l} \ln x = u \\ \dfrac{dx}{x} = du \\ v = \int x dx;\ v = \dfrac{x^2}{2} \end{array} \right| = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{dx}{x}$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$$

3. $$\int x^n \ln x\, dx = \left| \begin{array}{l} \ln x = u \\ \dfrac{dx}{x} = du \\ v = \int x^n dx;\ \ v = \dfrac{x^{n+1}}{n+1} \end{array} \right| = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1} \cdot \frac{dx}{x}$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n dx = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\,\frac{x^{n+1}}{(n+1)}$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C.$$

4. $$\int x^4 \ln^3 x\,dx = \left| \begin{matrix} \ln^3 x = u \\ \dfrac{3\ln^2 x}{x}dx = du \\ v = \int x^4 dx = \dfrac{x^5}{5} \end{matrix} \right| = \frac{x^5}{5}\ln^3 x - \int \frac{3x^5}{5}\cdot\frac{\ln^2 x}{x}dx$$

$$= \frac{x^5}{5}\ln^3 x - \frac{3}{5}\int x^4 \ln^2 x dx = \frac{x^5}{5}\ln^3 x - \frac{3}{5}\cdot I_1,$$

ku

$$I_1 = \int x^4 \ln^2 dx = \left| \begin{matrix} \ln^2 x = u \\ \dfrac{2\ln x}{x}dx = du \\ v = \dfrac{x^5}{5} \end{matrix} \right| = \frac{x^5}{5}\ln^2 x - \int \frac{x^5}{5}\frac{2\ln x}{x}dx$$

$$= \frac{x^5}{5}\ln^2 x - \frac{2}{5}\int x^4 \ln x = \frac{x^5}{5}\ln^2 x - \frac{2}{5}I_2.$$

Në bazë të detyrës 3.

$$I_2 = \int x^4 \ln x dx = \frac{x^5}{5}\ln x - \frac{x^5}{5^2}$$

Pra $I_1 = \dfrac{x^5}{5}\ln^2 x - \dfrac{2}{5}\left(\dfrac{x^5}{5}\ln x - \dfrac{x^5}{5^2}\right) = \dfrac{x^5}{5}\ln^2 x - \dfrac{2}{25}x^5 \ln x + \dfrac{2}{125}x^5.$

Përfundimisht,

$$I = \frac{x^5}{5}\ln^3 x - \frac{3}{5}\left(\frac{x^5}{5}\ln^2 x - \frac{2}{25}x^5 \ln x + \frac{2}{125}x^5\right) + C.$$

Detyra për ushtrime

Të njehsohen integralet:

1. $\int x^2 \ln x\, dx.$ **2.** $\int x \ln^2 x\, dx.$ **3.** $\int x^3 \ln^2 x\, dx.$

Të njehsohen integralet:

5. $\int \frac{\ln^2 x}{x^2} dx.$ 6. $\int (x^2 + x)\ln(x+1)\,dx.$

7. $\int \ln(x + \sqrt{1+x^2})dx.$ **8.** $\int x^2 \ln\frac{1-x}{1+x}dx.$

Zgjidhja.

5. $$\int \frac{\ln^2 x}{x^2}dx = \left| \begin{array}{l} \ln^2 x = u \\ 2\ln x \cdot \frac{1}{x}dx = du \\ v = \int x^{-2}dx = -x^{-1} \end{array} \right| = -\frac{1}{x}\ln^2 x + \int \frac{1}{x}\cdot 2\ln x \cdot \frac{1}{x}dx$$

$$= -\frac{1}{x}\ln^2 x + 2\int \frac{\ln x}{x^2}dx = -\frac{1}{x}\ln^2 x + 2I_1,$$

ku

$$I_1 = \int \frac{\ln x}{x^2}dx = \left| \begin{array}{l} \ln x = u \\ \frac{dx}{x} = du \\ v = -x^{-1} \end{array} \right| = -x^{-1}\ln x + \int \frac{1}{x}\frac{dx}{x} = -\frac{1}{x}\ln x + \int x^{-2}dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x}.$$

Pra,

$$I = -\frac{1}{x}\ln^2 x + 2\left(-\frac{1}{x}\ln x - \frac{1}{x}\right) = -\frac{1}{x}(\ln^2 x + 2\ln x + 2) + C.$$

6. $$\int (x^2+x)\ln(x+1)dx = \left| \begin{array}{l} \ln(x+1) = u \\ \frac{dx}{x+1} = du \\ v = \frac{x^3}{3} + \frac{x^2}{2} \end{array} \right| = \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\ln(x+1)$$

$$-\int \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\frac{dx}{x+1} = \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\ln(x+1) - \int \frac{2x^3+3x^2}{6(x+1)}dx$$

$$=\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\ln(x+1)-I_1,$$

ku

$$I_1=\int\frac{2x^3+3x^2}{6(x+1)}dx=\int\frac{2x^3+2x^2+x^2}{6(x+1)}dx=\frac{2}{6}\int\frac{x^2(x+1)}{x+1}dx$$

$$+\frac{1}{6}\int\frac{x^2}{x+1}dx=\frac{1}{3}\int x^2\,dx+\frac{1}{6}\int\left(x-1+\frac{1}{x+1}\right)dx$$

$$=\frac{x^3}{9}+\frac{1}{6}\left(\frac{x^2}{2}-x+\ln(x+1)\right).$$

Pra

$$I=\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\ln(x+1)--\frac{x^3}{9}-\frac{x^2}{12}+\frac{x}{6}-\frac{1}{6}\ln(x+1)$$

$$=\left(\frac{x^3}{3}+\frac{x^2}{2}-\frac{1}{6}\right)\ln(x+1)-\frac{x^3}{9}-\frac{x^2}{12}+\frac{x}{6}+C.$$

7. $$\int\ln(x+\sqrt{1+x^2})dx=\left|\begin{array}{l}\ln(x+\sqrt{1+x^2})=u\\ \dfrac{(x+\sqrt{1+x^2})'}{x+\sqrt{1+x^2}}dx=du\\ \dfrac{1+\dfrac{1}{\sqrt{x^2+1}}}{x+\sqrt{1+x^2}}dx=du\\ \dfrac{dx}{\sqrt{x^2+1}}=du\\ v=\int dx\Rightarrow v=x\end{array}\right|$$

$$=x\ln(x+\sqrt{1+x^2})-\int\frac{xdx}{\sqrt{x^2+1}}=x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}+C.$$

8. $\int x^2 \ln\frac{1-x}{1+x}dx = \left| \begin{matrix} \ln\frac{1-x}{1+x} = u \\ \frac{2}{x^2-1}dx = du \\ v = \frac{x^3}{3} \end{matrix} \right| = \frac{x^3}{3}\ln\frac{1-x}{1+x} - \int \frac{x^3}{3}\cdot\frac{2}{x^2-1}dx$

$$= \frac{x^3}{3}\ln\frac{1-x}{1+x} - \frac{2}{3}\int\left(x + \frac{x}{x^2-1}\right)dx = \frac{x^3}{3}\ln\frac{1-x}{1+x} - \frac{2}{3}\cdot\frac{x^2}{2}$$

$$-\frac{2}{3}\cdot\frac{1}{2}\ln|x^2-1| = \frac{x^3}{3}\ln\frac{1-x}{1+x} - \frac{x^2}{3} - \frac{1}{3}\ln|x^2-1| + C.$$

Detyra për ushtrime

Të njehsohen integralet:

4. $\int \sqrt{x}\ln^2 x\,dx.$ **5.** $\int\left(\frac{\ln x}{x}\right)^2 dx.$ **6.** $\int \frac{\ln x}{x^3}dx.$

Të njehsohen integralet:

9. $\int xe^{-x}dx.$ **10.** $\int x^3 e^{-x^2}dx.$ **11.** $\int e^{\sqrt{x}}dx.$ **12.** $\int e^{x+\ln x}dx.$

Zgjidhja.

9. $\int xe^{-x}dx = \left| \begin{matrix} x = u \\ dx = du \\ v = \int e^{-x}dx \\ v = -e^{-x} \end{matrix} \right| = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C.$

10. $\int x^3 e^{-x^2}dx = \int x^2\cdot x\cdot e^{-x^2}dx = \left| \begin{matrix} x^2 = u \\ 2xdx = du \\ v = \int xe^{-x^2}dx \\ v = -\frac{1}{2}e^{-x^2} \end{matrix} \right| = -\frac{1}{2}x^2 e^{-x^2} + \frac{1}{2}\int 2xe^{-x^2}dx$

$$= -\frac{1}{2}x^2 e^{-x^2} + \int xe^{-x^2}dx = -\frac{1}{2}x^2 e^{-x^2} - \frac{1}{2}e^{-x^2} = -\frac{1}{2}e^{-x^2}(x^2+1) + C.$$

11. $\int e^{\sqrt{x}}dx = \int \sqrt{x}\,\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \left| \begin{array}{l} \sqrt{x} = u \Rightarrow \frac{1}{2\sqrt{x}}dx = du \\ v = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \left| \begin{array}{l} \sqrt{x} = t \\ \frac{dx}{2\sqrt{x}} = dt \\ \frac{dx}{\sqrt{x}} = 2dt \end{array} \right| = 2\int e^t dt = 2e^{\sqrt{x}} \end{array} \right|$

$$= 2\sqrt{x}e^{\sqrt{x}} - \int 2e^{\sqrt{x}}\,\frac{dx}{2\sqrt{x}} = 2\sqrt{x}e^{\sqrt{x}} - \int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}}$$

$$= 2(\sqrt{x} - 1)e^{\sqrt{x}}.$$

12. $\int e^{x+\ln x}dx = \int e^x e^{\ln x}dx = \int xe^x dx = \left| \begin{array}{l} x = u, dx = du \\ v = e^x \end{array} \right| = xe^x - \int e^x dx$

$$= xe^x - e^x = e^x(x-1) + C.$$

Detyra për ushtrime

Të njehsohen integralet:

7. $\int x^2 e^{-x}dx.$ **8.** $\int x^n e^{-x^{n-1}}dx.$ **9.** $\int e^{\sqrt{2x+1}}dx.$

10. $\int e^{x+3\ln x}dx.$ **11.** $\int \frac{x^2}{e^x}dx.$ **12.** $\int x^2 \cdot 3^x dx.$

Të njesohen integralet

13. $\int e^{2x}(x^2 + 3x - 4)\,dx.$ **14.** $\int (x^2 + x + 1)\sin x\,dx.$

15. $\int x\sin 2x\,dx.$ **16.** $\int x^2\cos 5x\,dx.$

Zgjidhja.

13. $\int e^{2x}(x^2 + 3x - 4)dx = \left| \begin{array}{l} x^2 + 3x - 4 = u \\ (2x+3)dx = du \\ v = \int e^{2x}dx \Rightarrow v = \frac{1}{2}e^{2x} \end{array} \right| = \frac{1}{2}e^{2x}(x^2 + 3x - 4)$

$$-\frac{1}{2}\int e^{2x}(2x+3)dx = \frac{1}{2}e^{2x}(x^2+3x-4) - \frac{1}{2}I_1.$$

$$I_1 = \int e^{2x}(2x+3)dx = \left| \begin{matrix} 2x+3=u \\ 2dx = du \\ v = \frac{1}{2}e^{2x} \end{matrix} \right| = \frac{1}{2}e^{2x}(2x+3) + \int 2\cdot\frac{1}{2}e^{2x}dx$$

$$= \frac{1}{2}e^{2x}(2x+3) + \int e^{2x}dx = \frac{1}{2}e^{2x}(2x+3) + \frac{1}{2}e^{2x}.$$

Përfundimisht,

$$I = \frac{1}{2}e^{2x}(x^2+3x-4) - \frac{1}{4}e^{2x}(2x+3) - \frac{1}{4}e^{2x}$$

$$= \frac{1}{4}e^{2x}(2x^2+6x-8-2x-3-1) = \frac{1}{2}e^{2x}(x^2+2x-6)+C.$$

14. $$\int (x^2+x+1)\sin x dx = \left| \begin{matrix} x^2+x+1=u \\ (2x+1)dx = du \\ v = \int \sin x dx \\ v = -\cos x \end{matrix} \right| = -\cos x(x^2+x+1)$$

$$+\int \cos x(2x+1)dx = -(x^2+x+1)\cos x + I_1,$$

ku

$$I_1 = \int (2x+1)\cos x dx = \left| \begin{matrix} 2x+1=u \\ 2dx = du \\ v = \int \cos x dx \\ v = \sin x \end{matrix} \right| = (2x+1)\sin x - 2\int \sin x dx$$

$$= (2x+1)\sin x + 2\cos x.$$

Pra, $I = -(x^2+x+1)\cos x + (2x+1)\sin x + 2\cos x$

$$= (1-x^2-x)\cos x + (2x+1)\sin x + C.$$

15. $\int x\sin 2x dx = \left| \begin{array}{l} x = u \\ dx = du \\ v = \int \sin 2x dx \\ v = -\frac{1}{2}\cos 2x \end{array} \right| = -\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x dx$

$$= -\frac{x\cos 2x}{2} + \frac{1}{4}\sin 2x + C.$$

16. $\int x^2 \cos 5x \, dx = \left| \begin{array}{l} x^2 = u \\ 2xdx = du \\ v = \frac{1}{5}\sin 5x \end{array} \right| = \frac{x^2 \sin 5x}{5} - \frac{2}{5}\int x \sin 5x \, dx$

$$= \frac{x^2 \sin 5x}{5} - \frac{2}{5} I_1, \text{ ku}$$

$$I_1 = \int x\sin 5x dx = \left| \begin{array}{l} x = u \\ dx = du \\ v = -\frac{1}{5}\cos 5x \end{array} \right| = -\frac{1}{5}x\cos 5x + \frac{1}{5}\int \cos 5x dx$$

$$= -\frac{1}{5}x\cos 5x + \frac{1}{25}\sin 5x.$$

Pra,

$$I = \frac{x^2 \sin 5x}{5} - \frac{2}{5}\left(-\frac{1}{5}x\cos 5x + \frac{1}{25}\sin 5x\right)$$

$$= \frac{x^2 \sin 5x}{5} + \frac{2}{25}x\cos 5x - \frac{2}{125}\sin 5x$$

$$= \left(\frac{x^2}{5} - \frac{2}{125}\right)\sin 5x + \frac{2}{25}x\cos 5x + C.$$

Detyra për ushtrime

Të njehsohen integralet:

13. $\int x^2 \sin 3x\, dx.$ **14.** $\int (x+1)\cos(x+2)\, dx.$

15. $\int (x^2 - x - 1)\cos 2x\, dx.$ **16.** $\int x^{-x}(x^2 - x + 3)dx.$

Të njehsohen integralet

17. $\int e^{-x}\cos x\, dx.$ **18.** $\int \sin(\ln x)\, dx.$ **19.** $\int \cos(\ln x)\, dx.$

20. $\int e^{x}\sin x\, dx.$ **21.** $\int e^{x}\cos x\, dx.$ **22.** $\int e^{x}\sin^2 x\, dx.$

23. $\int e^{x}\cos^2 x\, dx.$

Zgjidhja.

17. $$\int e^{-x}\cos x\, dx = \begin{vmatrix} \cos x = u \\ -\sin x dx = du \\ v = -e^{-x} \end{vmatrix} = -e^{-x}\cos x - \int e^{-x}\sin x\, dx$$
$$= -e^{-x}\cos x - I_1,$$

ku

$$I_1 = \int e^{-x}\sin x\, dx = \begin{vmatrix} \sin x = u \\ \cos x dx = du \\ v = -e^{-x} \end{vmatrix} = -e^{-x}\sin x + \int e^{-x}\cos x\, dx$$
$$= -e^{-x}\sin x + I.$$

Pra,

$$I = -e^{-x}\cos x - (-e^{-x}\sin x + I)$$
$$I = -e^{-x}\cos x + e^{-x}\sin x - I$$
$$2I = e^{-x}(\sin x - \cos x)$$
$$I = \frac{e^{-x}}{2}(\sin x - \cos x) + C.$$

18. $\int \sin(\ln x)dx.$ **19.** $\int \cos(\ln x)dx.$

Në vijim, njëkohësisht do të zgjidhim integralet në detyrat 18, 19.
Le të shënojmë:

$$I_1 = \int \sin(\ln x)dx; \qquad I_2 = \int \cos(\ln x)dx.$$

Zgjidhim I_1. Merret:

$$I_1=\int \sin(\ln x)dx=\left|\begin{array}{l}\sin(\ln x)=u\\ \dfrac{\cos(\ln x)}{x}dx=du\\ v=x\end{array}\right|=x\sin(\ln x)-\int x\frac{\cos(\ln x)}{x}dx$$

$$=x\sin(\ln x)-\int \cos(\ln x)dx=x\sin(\ln x)-I_2.$$

Pra, kemi:

$I_1=x\sin(\ln x)-I_2$ gjegjësisht

$I_1+I_2=x\sin(\ln x)$ (1)

Zgjidhim I_2. Merret:

$$I_2=\int \cos(\ln x)dx=\left|\begin{array}{l}\cos(\ln x)=u\\ \dfrac{-\sin(\ln x)}{x}dx=du\\ v=x\end{array}\right|=x\cos\ln x+\int x\frac{\sin(\ln x)}{x}dx$$

$$=x\cos(\ln x)+\int \sin(\ln x)\,dx=x\cos(\ln x)+I_1.$$

Pra,

$I_2=x\cos(\ln x)+I_1$, gjegjësisht,

$-I_1+I_2=x\cos(\ln x).$ (2)

Nga (1), (2) merret sistemi:

$$\begin{cases}I_1+I_2=x\sin(\ln x)\\ -I_1+I_2=x\cos(\ln x)\end{cases}$$

me zgjidhjen e të cilit merret:

$$I_1=\frac{x}{2}(\sin(\ln x)-\cos(\ln x)),\quad I_2=\frac{x}{2}(\sin(\ln x)+\cos(\ln x)).$$

20. $\int e^x\sin x dx.$ **21.** $\int e^x\cos x dx.$

Veprojmë ngjashëm si në detyrat paraprake.

Shënojmë: $I_1=\int e^x\sin x dx,\qquad I_2=\int e^x\cos x dx.$

Zgjidhim integralin I_1.

$$I_1 = \int e^x \sin x dx = \begin{vmatrix} \sin x = u \\ \cos x dx = du \\ v = e^x \end{vmatrix} = e^x \sin x - \int e^x \cos x dx = e^x \sin x - I_2.$$

Pra, $I_1 + I_2 = e^x \sin x$. (1)

Zgjidhim integralin I_2.

$$I_2 = \int e^x \cos x dx = \begin{vmatrix} \cos x = u \\ -\sin x dx = du \\ v = e^x \end{vmatrix} = e^x \cos x + \int e^x \sin x dx$$

$$= e^x \cos x + I_1.$$

Pra $-I_1 + I_2 = e^x \cos x$ (2)

Nga (1) dhe (2) merret sistemi:

$$\begin{cases} I_1 + I_2 = e^x \sin x \\ -I_1 + I_2 = e^x \cos x \end{cases}$$

me zgjidhjen e të cilit merret:

$$I_1 = \frac{e^x}{2}(\sin x - \cos x),\ I_2 = \frac{e^x}{2}(\sin x + \cos x).$$

Duke vepruar si më sipër tregoni se:

$$\int e^{ax} \sin bx dx = \frac{a \sin bx - b \cos bx}{a^2 + b^2} \cdot e^{ax}$$

$$\int e^{ax} \cos bx dx = \frac{a \cos bx + b \sin bx}{a^2 + b^2} \cdot e^{ax}.$$

22. $\int e^x \sin^2 x dx = \int e^x \cdot \frac{1 - \cos 2x}{2} dx = \frac{1}{2}\int e^x dx - \frac{1}{2}\int e^x \cos 2x dx$

$$= \frac{1}{2}e^x - \frac{1}{2}\left(\frac{\cos 2x + 2\sin 2x}{5}\right)e^x + C.$$

Shënim. Shprehja e fundit mund të transformohet në

$$\frac{e^x}{5}(\sin^2 x - 2\sin x \cos x + 2) + C.$$

23. $\int e^x \cos^2 x dx = \int e^x \cdot \frac{1 + \cos 2x}{2} dx = \frac{1}{2}\int e^x dx + \frac{1}{2}\int e^x \cos 2x dx$

$$=\frac{1}{2}e^x+\frac{1}{2}\left(\frac{\cos 2x+2\sin 2x}{5}\right)e^x$$

$$=\frac{1}{2}e^x\left(1+\frac{\cos^2 x-\sin^2 x+4\sin x\cos x}{5}\right)$$

$$=\frac{1}{5}e^x(\cos^2 x+2\sin x\cos x+2)+C.$$

Detyra për ushtrime

Të njehsohen integralet:

17. $\int e^{2x}\cos 3x\,dx.$ **18.** $\int e^{2x}\sin 3x\,dx.$ **19.** $\int \cos^2(\ln x)\,dx.$

20. $\int \sin^2(\ln x)\,dx.$ **21.** $\int e^{ax}\sin^2 x\,dx.$ **22.** $\int e^{ax}\cos^2 x\,dx.$

23. $\int e^{x}\sin^4 x\,dx.$ **24.** $\int e^{x}\cos^4 x\,dx.$

Të njehsohen integralet

24. $\int \sqrt{x^2-a^2}\,dx.$ **25.** $\int \sqrt{x^2+a^2}\,dx.$ **26.** $\int \arcsin x\,dx.$

27. $\int \arctan x\,dx.$ **28.** $\int x^2\arccos x\,dx.$ **29.** $\int (\arcsin x)^2\,dx.$

30. $\int x\,\mathrm{arc\,tg}x\,dx.$ **31.** $\int x(\mathrm{arc\,tg}x)^2\,dx.$ **32.** $\int \frac{x^2}{(1+x^2)^2}\,dx.$

33. $\int \frac{dx}{(1+x^2)^2}.$ **34.** $\int x^2\sqrt{1+x^2}\,dx.$ **35.** $\int \frac{x^5}{\sqrt{1-x^4}}\,dx.$

Zgjidhja.

24. $$I=\int\sqrt{x^2-a^2}\,dx=\left|\begin{array}{l}\sqrt{x^2-a^2}=u\\ \dfrac{xdx}{\sqrt{x^2-a^2}}=du\\ v=x\end{array}\right|=x\cdot\sqrt{x^2-a^2}-\int x\cdot\frac{xdx}{\sqrt{x^2-a^2}}$$

$$=x\sqrt{x^2-a^2}-\int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx$$

$$=x\sqrt{x^2-a^2}-\left(\int\frac{x^2-a^2}{\sqrt{x^2-a^2}}dx-a^2\int\frac{dx}{\sqrt{x^2-a^2}}\right)$$

$$=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}\,dx$$

$$-a^2\int \frac{dx}{\sqrt{x^2-a^2}} = x\sqrt{x^2-a^2} - I - a^2 \ln|a+\sqrt{x^2-a^2}|.$$

Pra

$$2I = a\sqrt{x^2-a^2} - a^2\ln|x+\sqrt{x^2-a^2}|$$

$$I = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C.$$

25. Duke vepruar ngjashëm si në detyrën paraprake merret:

$$\int\sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}| + C.$$

26. $$\int \arcsin x dx = \left| \begin{array}{l} \arcsin x = u \\ \dfrac{1}{\sqrt{1-x^2}}dx = du \\ v = x \end{array} \right| = x\arcsin x - \int \frac{xdx}{\sqrt{1-x^2}}$$

$$= x\arcsin x + \sqrt{1-x^2} + C.$$

27. $$\int \arctan x dx = \left| \begin{array}{l} \arctan x = u \\ \dfrac{dx}{1+x^2} = du \\ v = x \end{array} \right| = x\arctan x - \int \frac{xdx}{1+x^2}$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

28. $$I = \int x^2 \arccos x dx = \left| \begin{array}{l} \arccos x = u,\ -\dfrac{1}{\sqrt{1-x^2}}dx = du \\ v = \dfrac{x^3}{3} \end{array} \right|$$

$$= \frac{x^3}{3}\arccos x + \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}dx = \frac{x^3}{3}\arccos x + \frac{1}{3}I_1, \text{ ku}$$

$$I_1 = \int x^2 \frac{x}{\sqrt{1-x^2}}dx = \left| \begin{array}{l} x^2 = u \\ 2xdx = du \\ v = \displaystyle\int \frac{xdx}{\sqrt{1-x^2}} \\ v = -\sqrt{1-x^2} \end{array} \right| = -x^2\sqrt{1-x^2} + \int \sqrt{1-x^2}\cdot 2dx$$

$$= -x^2\sqrt{1-x^2} - \frac{2}{3}\sqrt{(1-x^2)^3}.$$

Pra

$$I = \frac{x^3}{3}\arccos x + \frac{1}{3}\left(-x^2\sqrt{1-x^2} - \frac{2}{3}\sqrt{(1-x^2)^3}\right) + C.$$

29. $$\int (\arcsin x)^2 dx = \left| \begin{array}{l} (\arcsin x)^2 = u \\ \dfrac{2\arcsin x}{\sqrt{1-x^2}} dx = du \\ v = x \end{array} \right|$$

$$= x(\arcsin x)^2 - 2\int x\frac{\arcsin x}{\sqrt{1-x^2}}dx = x(\arcsin x)^2 - 2I_1,$$

ku

$$I_1 = \int \arcsin x \cdot \frac{x}{\sqrt{1-x^2}}dx = \left| \begin{array}{l} \arcsin x = u, \dfrac{dx}{\sqrt{1-x^2}} = du \\ v = \int \dfrac{x}{\sqrt{1-x^2}}dx \Rightarrow v = -\sqrt{1-x^2} \end{array} \right|$$

$$= -\sqrt{1-x^2}\cdot\arcsin x + \int\sqrt{1-x^2}\cdot\frac{dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\cdot\arcsin x + x.$$

Pra,

$$I = x(\arcsin x)^2 + 2\sqrt{1-x^2}\cdot\arcsin x - 2x + C.$$

30. $$\int x\arctan x\, dx = \left| \begin{array}{l} \arctan x = u \\ \dfrac{dx}{1+x^2} = du \\ v = \dfrac{x^2}{2} \end{array} \right| = \frac{x^2}{2}\cdot\arctan x - \int\frac{x^2}{2}\cdot\frac{dx}{1+x^2}$$

$$= \frac{x^2}{2}\cdot\arctan x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\cdot\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x = \frac{x^2+1}{2}\cdot\arctan x - \frac{1}{2}x + C.$$

31. $I=\int x(\arctan x)^2\,dx=\left|\begin{array}{l}(\arctan x)^2=u\\ \dfrac{2\arctan x}{1+x^2}dx=du\\ v=\dfrac{x^2}{2}\end{array}\right|$

$$=\frac{x^2}{2}(\arctan x)^2-\int\frac{x^2\arctan x}{1+x^2}dx=\frac{x^2}{2}(\arctan x)^2-I_1,$$

ku $I_1=\int\arctan x\cdot\frac{x^2}{1+x^2}dx=\left|\begin{array}{l}\arctan x=u,\ \dfrac{1}{1+x^2}dx=du\\ v=\displaystyle\int\frac{x^2}{1+x^2}dx\Rightarrow v=x-\arctan x\end{array}\right|$

$$=(x-\arctan x)\arctan x-\int\frac{(x-\arctan x)}{1+x^2}dx$$

$$=x\arctan x-(\arctan x)^2-\frac{1}{2}\int\frac{x}{1+x^2}dx+\int\frac{\arctan x}{1+x^2}dx$$

$$=x\arctan x-(\arctan x)^2-\frac{1}{2}\ln(1+x^2)+\frac{(\arctan x)^2}{2}$$

$$=x\arctan x-\frac{1}{2}(\arctan x)^2-\frac{1}{2}\ln(1+x^2).$$

Pra

$$I=\frac{x^2}{2}(\arctan x)^2-x\arctan x+\frac{1}{2}(\arctan x)^2+\frac{1}{2}\ln(1+x^2)$$

$$=\frac{x^2+1}{2}(\arctan x)^2-x\arctan x+\frac{1}{2}\ln(1+x^2)+C.$$

32. $\int\frac{x^2}{(1+x^2)^2}dx=\int x\cdot\frac{xdx}{(1+x^2)^2}=\left|\begin{array}{l}x=u\\ dx=du\\ v=\displaystyle\int\frac{xdx}{(1+x^2)^2}\\ v=-\dfrac{1}{2}\cdot\dfrac{1}{1+x^2}\end{array}\right|=-\frac{1}{2}\frac{x}{1+x^2}+\frac{1}{2}\int\frac{dx}{1+x^2}$

$$=-\frac{1}{2}\frac{x}{1+x^2}+\frac{1}{2}\arctan x+C.$$

33. $I=\int\frac{dx}{(1+x^2)^2}=\int\frac{1+x^2-x^2}{(1+x^2)^2}dx=\int\frac{1+x^2}{(1+x^2)^2}dx-\int\frac{x^2}{(1+x^2)^2}dx$

$$= \int \frac{dx}{1+x^2} - \int \frac{x^2}{(1+x^2)^2}\,dx.$$

Integrali $\int \frac{x^2}{(1+x^2)^2}\,dx$ e zgjidhëm në detyrën paraprake, prandaj

$$I = \arctan x + \frac{1}{2}\frac{x}{1+x^2} - \frac{1}{2}\arctan x = \frac{\arctan x}{2} + \frac{1}{2}\frac{x}{1+x^2} + C.$$

34. $$I = \int x^2\sqrt{1+x^2}\,dx = \int x \cdot x \cdot \sqrt{1+x^2}\,dx = \left| \begin{array}{l} x = u \\ dx = du \\ v = \int x\sqrt{1+x^2}\,dx \\ v = \frac{(1+x^2)\sqrt{1+x^2}}{3} \end{array} \right|$$

$$= \frac{x}{3}(1+x^2)\sqrt{1+x^2} - \frac{1}{3}\int (1+x^2)\sqrt{1+x^2}\,dx$$

$$= \frac{x}{3}(1+x^2)\sqrt{1+x^2} - \frac{1}{3}\int (1+x^2)dx - \frac{1}{3}\int x^2\sqrt{1+x^2}\,dx$$

$$= \frac{x}{3}(1+x^2)\sqrt{1+x^2} - \frac{1}{3}I_1 - \frac{1}{3}I,$$

ku

$$I_1 = \int \sqrt{1+x^2}\,dx = \frac{x}{2}\sqrt{1+x^2} - \frac{1}{2}\ln|x+\sqrt{1+x^2}|.$$

Pra

$$I + \frac{1}{3}I = \frac{x}{3}(1+x^2)\sqrt{1+x^2} - \frac{1}{3}\frac{x}{2}\sqrt{1+x^2} - \frac{1}{6}\ln|x+\sqrt{1+x^2}|$$

$$I = \frac{x(1+2x^2)}{8}\sqrt{1+x^2} - \frac{1}{8}\ln|x+\sqrt{1+x^2}| + C.$$

35. $$I = \int \frac{x^5}{\sqrt{1-x^4}}\,dx = \int x^2 \cdot \frac{x^3}{\sqrt{1-x^4}}\,dx = \left| \begin{array}{l} x^2 = u \\ 2xdx = du \\ v = \int \frac{x^3}{\sqrt{1-x^4}}\,dx \\ v = -\frac{1}{2}\sqrt{1-x^4} \end{array} \right|$$

$$= -\frac{x^2}{2}\sqrt{1-x^4} + \frac{1}{2}\cdot 2\int x\sqrt{1-x^4}\,dx = -\frac{x^2}{2}\sqrt{1-x^4} + I_1.$$

$$I_1 = \int x\sqrt{1-x^4}\,dx = \int x\sqrt{1-(x^2)^2}\,dx = \begin{vmatrix} x^2 = t \\ xdx = \dfrac{dt}{2} \end{vmatrix} = \frac{1}{2}\int \sqrt{1-t^2}\,dt$$

$$= \frac{1}{2}\left(\frac{1}{2}\arcsin t + \frac{t}{2}\sqrt{1-t^2}\right) = \frac{1}{4}\arcsin x^2 + \frac{x^2}{4}\sqrt{1-x^4}.$$

Pra,

$$I = -\frac{x^2}{2}\sqrt{1-x^4} + \frac{1}{4}\arcsin x^2 + \frac{x^2}{4}\sqrt{1-x^4}$$

$$= -\frac{x^2}{4}\sqrt{1-x^4} + \frac{\arcsin x^2}{4} + C.$$

Detyra për ushtrime

Të njehsohen integralet:

25. $\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx.$

26. $\int \arcsin x \cdot \arccos x\,dx.$

27. $\int \frac{x^2 e^x}{(x+2)^2}\,dx.$

28. $\int \frac{x^2 \arctan x}{1+x^2}\,dx.$

29. $\int \frac{dx}{(a^2+x^2)^3}.$

30. $\int x^2\sqrt{a^2+x^2}\,dx.$