

## 5. Integralet rekurente

Të gjenden formulat rekurente për njehsimin e integraleve:

$$1. \int x^n e^{-x} dx. \quad 2. \int x^a (\ln x)^n dx, \quad a \neq -1. \quad 3. \int \sin^n x dx.$$

$$4. \int \frac{dx}{\sin^n x}. \quad 5. \int \frac{dx}{(x^2 + a^2)^n}, \quad n \neq 1.$$

**Zgjidhja.**

$$1. I_n = \int x^n e^{-x} dx = \left| \begin{array}{l} x^n = n \\ nx^{n-1} dx = du \\ v = -e^{-x} \end{array} \right| = -x^n e^{-x} - \int -e^{-x} nx^{n-1} dx$$

$$= -x^n e^{-x} + n \int x^{n-1} e^{-x} dx = -x^n e^{-x} + nI_{n-1}.$$

$$2. \int x^a (\ln x)^n dx = \left| \begin{array}{l} (\ln x)^n = u \\ n(\ln x)^{n-1} dx = du \\ v = x^{a+1} \end{array} \right|$$

$$= \frac{x^{a+1}}{a+1} (\ln x)^n - \frac{n}{a+1} \int \frac{x^{a+1} (\ln x)^{n-1}}{x} dx$$

$$= \frac{x^{a+1}}{a+1} (\ln^n x) - \frac{n}{a+1} \int x^a (\ln^{n-1} x) dx = \frac{x^{a+1}}{a+1} \ln^n x - \frac{n}{a+1} I_{n-1}.$$

$$3. I_n = \int \sin^n x dx = \int \sin^{n-1} x \sin x dx = \left| \begin{array}{l} \sin^{n-1} x = u \\ (n-1) \sin^{n-2} x \cdot \cos x dx = du \\ v = -\cos x \end{array} \right|$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cos^2 x dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x (1 - \sin^2) dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x dx - (n-1)I_n.$$

Pra

$$I_n + (n-1)I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2}$$

prej nga merret:

$$I_n = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}.$$

$$4. I_n = \int \frac{dx}{\sin^n x} = \int \frac{1}{\sin^{n-2} x} \cdot \frac{dx}{\sin^2 x} = \left. \begin{array}{l} u = \frac{1}{\sin^{n-2} x} \\ du = -\frac{(n-2)\cos x}{\sin^{n-1} x} dx \\ v = \int \frac{dx}{\sin^2 x} = -\operatorname{ctgx} \end{array} \right|$$

$$= -\frac{\operatorname{ctgx}}{\sin^{n-2} x} - (n-2) \int \operatorname{ctgx} \frac{\cos x}{\sin^{n-1} x} dx$$

$$= -\frac{\frac{\cos x}{\sin x}}{\sin^{n-2} x} - (n-2) \int \frac{\cos x}{\sin x} \frac{\cos x}{\sin^{n-1} x} dx$$

$$= -\frac{\cos x}{\sin^{n-1} x} - (n-2) \int \frac{\cos^2 x}{\sin^n x} dx$$

$$= -\frac{\cos x}{\sin^{n-1} x} - (n-2) \int \frac{1 - \sin^2 x}{\sin^n x} dx$$

$$= -\frac{\cos x}{\sin^{n-1} x} - (n-2) \left[ \int \frac{dx}{\sin^n x} - \int \frac{dx}{\sin^{n-2} x} \right]$$

$$= -\frac{\cos x}{\sin^{n-1} x} - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n + (n-2)I_n = -\frac{\cos x}{\sin^{n-1} x} + (n-2)I_{n-2}$$

$$I_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}, \quad n \geq 2.$$

$$\begin{aligned}
5. \quad I_n &= \int \frac{dx}{(x^2 + a^2)^n} = \frac{1}{a^2} \int \frac{x^2 + a^2 - x^2}{(x^2 + a^2)^n} dx \\
&= \frac{1}{a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}} - \frac{1}{a^2} \int x \frac{xdx}{(x^2 + a^2)^n} \\
&= \frac{1}{a^2} I_{n-1} - \frac{1}{a^2} \int x \frac{xdx}{(x^2 + a^2)^n} \tag{1}
\end{aligned}$$

Le të njehsojmë  $\int x \frac{xdx}{(x^2 + a^2)^n}$ . Kemi zëvendësimin

$$x = u$$

$$dx = du$$

$$v = \int \frac{xdx}{(x^2 + a^2)} = -\frac{1}{2(n-1)(a^2 + x^2)^{n-1}}$$

Pra

$$\begin{aligned}
\int \frac{x^2 dx}{(x^2 + a^2)^n} &= -\frac{x}{2(n-1)(a^2 + x^2)^{n-1}} + \frac{1}{2(n-1)} \int \frac{dx}{(a^2 + x^2)^{n-1}} \\
&= -\frac{x}{2(n-1)(a^2 + x^2)^{n-1}} + \frac{1}{2(n-1)} I_{n-1}.
\end{aligned}$$

Pra, duke zëvendësuar rezultatin e fundit në (1) merret:

$$\begin{aligned}
I_n &= \frac{1}{a^2} I_{n-1} - \frac{1}{a^2} \left( -\frac{x}{2(n-1)(a^2 + x^2)^{n-1}} + \frac{1}{2(n-1)} I_{n-1} \right) \\
&= \frac{1}{a^2} \left( 1 - \frac{1}{2(n-1)} \right) I_{n-1} + \frac{x}{2a^2(n-1)(a^2 + x^2)^{n-1}} \\
&= \frac{1}{2a^2(n-1)} \left( (2n-3)I_{n-1} + \frac{x}{(a^2 + x^2)^{n-1}} \right).
\end{aligned}$$

**Detyra për ushtrime**

Të njehsohen integralet:

$$1. I_n = \int e^{ax} \sin^n x dx = \frac{e^{ax}}{a^2 + n^2} \sin^{n-1} x (a \sin x - n \cos x) + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

$$2. I_n = \int \cos^4 x dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}.$$

$$3. a) I_n = \int \operatorname{tg}^n x dx = \frac{1}{a(n-1)} \operatorname{tg}^{n-1} x - I_{n-2}$$

$$b) I_n = \int \operatorname{ctg}^n x dx = -\frac{1}{(n-1)} \operatorname{ctg}^{n-1} x - I_{n-2}, (n \neq 1).$$

$$4. I_n = \int (a^2 - x^2)^n dx = \frac{x(a-x)^2}{2n+1} + \frac{2na^2}{2n+1} I_{n-1}, n \neq -\frac{1}{2}.$$

$$5. I_n = \int \frac{dx}{\cos^n x} = \frac{1}{n-1} \frac{\sin x}{\cos^{n-1} x} - \frac{n-2}{n-1} I_{n-2}.$$

$$6. I_n = \int \frac{xdx}{\sin^n ax} = -\frac{x \cos ax}{(n-1)a \sin^{n-1} x} - \frac{1}{(n-1)(n-2)a^2 \sin^{n-2} ax} + \frac{n-2}{n-1} I_{n-2}, (n > 2).$$

$$7. I_n = \int \frac{xdx}{\cos^n ax} = -\frac{x \sin ax}{(n-1)a \cos^{n-1} x} - \frac{1}{(n-1)(n-2)a^2 \cos^{n-2} x} + \frac{n-2}{n-1} I_{n-2}, (n > 2).$$

$$8. I_n = \int \frac{xdx}{\sin ax \cos^n ax} = \frac{1}{a(n-1) \cos^{n-1} ax} + I_{n-2}, (n \neq 1).$$

$$9. I_n = \int \frac{xdx}{\sin^n ax \cos ax} = -\frac{1}{a(n-1) \sin^{n-1} ax} + I_{n-2}, (n \neq 1).$$

$$10. I_n = \int \frac{\sin^n ax}{\cos ax} dx = -\frac{\sin^{n-1} ax}{a(n-1)} + I_{n-2}.$$

$$11. I_n = \int \frac{\cos^n ax}{\sin ax} dx = -\frac{\cos^{n-1} ax}{a(n-1)} + I_{n-2}, (n \neq 1).$$

$$12. I_n = \int x^n e^{ax} dx = \frac{x^n e^{ax}}{n} - \frac{n}{a} I_{n-1}.$$

$$13. I_n = \int \frac{e^{ax}}{x^n} dx = \frac{1}{n-1} \left( -\frac{e^{ax}}{x^{n-1}} + a I_{n-1} \right), \quad n \neq 1.$$

$$14. I_n = \int e^{ax} \cos^n x dx = \frac{e^{ax} \cos^{n-1} x}{a^2 + n^2} (a \cos x + n \sin x) \\ + \frac{n(n-1)}{a^2 + n^2} I_{n-2}, \quad n \neq -1.$$

$$15. I_n = \int \ln^n x dx = x \ln^n x - n I_{n-1}, \quad n \neq -1.$$

$$16. I_n = \int \frac{dx}{(\ln x)^n} = -\frac{x}{(n-1) \ln^{n-1} x} + \frac{1}{n-1} I_{n-1}, \quad n \neq 1.$$

$$17. I_n = \int \frac{\ln^n x}{x^m} dx = -\frac{\ln^n x}{(m-1)x^{m-1}} + \frac{n}{m-1} I_{n-1}, \quad m \neq 1.$$

$$18. I_n = \int \frac{x^m}{\ln^n x} dx = -\frac{x^{m+1}}{(m-1) \ln^{n-1} x} + \frac{m+1}{n-1} I_{n-1}, \quad n \neq 1.$$

$$19. I_n = \int \frac{dx}{x^m \ln^n x} = -\frac{1}{x^{m-1} (n-1) \ln^{n-1} x} - \frac{m-1}{n-1} I_{n-1}, \quad n \neq 1.$$