7. Integrimi i funksioneve trigonometrike

1) Le të jetë $\int R(\sin x, \cos x)dx,$ (1) ku R është funksion racional.

Në rastin e përgjithshëm merret zëvendësimi:

$$\tan\frac{x}{2} = t. \text{ Atëherë } \frac{x}{2} = \arctan t \Rightarrow x = 2\arctan t,\ dx = \frac{2dt}{1+t^2}.$$

$$\sin x = \frac{\sin 2\cdot\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{2t}{1+t^2}.$$

$$\cos x = \frac{1-t^2}{1+t^2}.\ \left(\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{4t^2}{(1+t^2)^2}} = \sqrt{\left(\frac{1-t^2}{1+t^2}\right)^2}\right).$$

Duke zëvendësuar në (1) merret:

$$\int R\left(\frac{2t}{1+t^2}.\frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

Pra, integrali i funksioneve trigonometrike shndërrohet në integral të funksionit racional.

2) Nëse funksioni integral plotëson kushtin

$R(-\sin x, \cos x) = -R(\sin x, \cos x),$ pra nëse funksioni integral është tek sipas sinx atëherë merret zëvendësimi: $\cos x = t;\ \sin x = \sqrt{1-t^2};$

$$x = \arcsin t;\ dx = -\frac{1}{\sqrt{1-t^2}}dt.$$

3) Nëse $R(\sin x, -\cos x) = -R(\sin x, \cos x),$ pra nëse funksioni nënintegral është tek sipas $\cos x,$ merret zëvendësimi:

$$\sin x = t;\ x = \arcsin t;\ dx = \frac{dt}{\sqrt{1-t^2}};\ \cos x = \sqrt{1-t^2}.$$

4) Nëse $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, pra nëse funksioni nënintegral është çift sipas $\sin x,\ \cos x,$ merret zëvendësimi:

$$\tan x = t;\ x = \arctan x;\ dx = \frac{dt}{1+t^2}.$$

Meqë

$$\sin^2 x = \frac{\sin^2 x}{\sin^2 x + \cos^2 x} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} = \frac{\tan^2 x}{\tan^2 x + 1} = \frac{t^2}{1+t^2}$$

atëherë

$$\sin x = \frac{t}{\sqrt{1+t^2}}$$

$$\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{t^2}{1+t^2}} = \frac{1}{\sqrt{1+t^2}}.$$

5) Integralet e tipit $\int \sin^m x \cos^n x dx$ shndërrohen në integrale të funksioneve racionale me këto zëvendësime:

i) Nëse m është numër tek, zëvendësojmë $\cos x = t$.

ii) Nëse n është numër tek, zëvendësojmë $\sin x = t$.

iii) Nëse m, n janë numra natyral çift zbatojmë formulat:

$$\sin^2 x = \frac{1-\cos 2x}{2};\ \cos^2 x = \frac{1+\cos 2x}{2}.$$

iv) Nëse m, n janë numra çift me shenja të kundërta, zëvendësojmë $\tan x = t$ ose $\mathrm{ctan} x = t$.

6) Integralet e tipit:

i) $\int \sin mx \cdot \sin nx dx.$

ii) $\int \sin mx \cdot \cos nx dx.$

iii) $\int \cos mx \cdot \cos nx dx.$

Zgjidhen duke zbatuar formuat, të cilat i kemi përmendur tek integrimi me metodën e zëvendësimit.

$$\sin mx\sin nx=\frac{1}{2}(\cos(m-n)x-\cos(m+n)x)$$

$$\sin mx\cos nx=\frac{1}{2}(\sin(m-n)x+\sin(m+n)x)$$

$$\cos mx\cos nx=\frac{1}{2}(\cos(m-n)x+\cos(m+n)x).$$

Të njehsohen integralet trigonometrike

1. $\int\frac{\cos^4 x}{\sin^3 x}dx.$ **2.** $\int\frac{\sin^3 x}{\cos^4 x}dx.$

3. $\int\sin^3 x\cos^4 xdx.$ **4.** $\int\text{ctan}^6 xdx.$

Zgjidhja.

1. Zëvendësojmë $\cos x=t;\sin x=\sqrt{1-t^2};dx=\frac{dt}{\sqrt{1-t^2}}$

$$\int\frac{\cos^4 x}{\sin^3 x}dx=\int\frac{t^4}{(1-t^2)^3}\frac{dt}{\sqrt{1-t^2}}=\int\frac{t^4}{(1-t^2)^2}dt$$

2. $\int\frac{\sin^3 x}{\cos^4 x}dx=\int\frac{\sin^2 x\sin x}{\cos^4 x}dx=\int\frac{1-\cos^2 x}{\cos^4 x}\sin xdx$

$$=\begin{vmatrix}\cos x=t\\ -\sin xdx=dt\\ \sin xdx=-dt\end{vmatrix}=-\int\frac{1-t^2}{t^4}dt=\frac{1}{3\cos^3 x}-\frac{1}{\cos x}+C.$$

3. Në këtë rast $m=3,$ zëvendësojmë $\cos x=t;\ -\sin xdx=dt$

$$\int\sin^3 x\cos^4 xdx=\int(1-\cos^2 x)\cos^4 x\sin xdx=-\int(1-t^2)t^4dt$$

$$= -\frac{t^5}{5} + \frac{t^7}{7} + C = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C.$$

4. $\int \operatorname{ctan}^6 x dx = \int \frac{\cos^6 x}{\sin^6 x} dx = \int \frac{\cos^4 x}{\sin^4 x} \cos^4 x \frac{dx}{\sin^2 x}$

$$= \int \operatorname{ctan}^4 x \cos^2 x \frac{dx}{\sin^2 x} = \left| \begin{array}{l} \operatorname{ctan} x = u \\ \frac{dx}{\sin^2 x} = -du \\ \cos^2 x = \frac{u^2}{1+u^2} \end{array} \right| = -\int \frac{u^6}{1+u^2} du.$$

Detyra për ushtrime

Të njehsohen integralet:

1. $\int \frac{\sin^6 x}{\cos^5 x} dx.$ **2.** $\int \frac{\sin^3 x}{\sqrt[3]{\cos^2 x}} dx.$ **3.** $\int \frac{dx}{\sin^3 \frac{x}{2} \sqrt{\cos^3 \frac{x}{2}}}.$

4. $\int \sin x \cos^2 x dx.$ **5.** $\int \frac{\sin^4 x}{\cos^6 x} dx.$

Duke i zbatuar formulat:

$$\sin(\alpha - \beta) = \sin((x+\alpha) - (x+\beta))$$

$$\cos(\alpha - \beta) = \cos((x+\alpha) - (x+\beta))$$

njesoni

5. $\int \frac{dx}{\sin(x+\alpha)\sin(x+\beta)}.$ **6.** $\int \frac{dx}{\sin x - \sin a}.$

Zgjidhja.

$$\int \frac{dx}{\sin(x+\alpha)\sin(x+\beta)} = \frac{1}{\sin(\alpha-\beta)} \int \frac{\sin((x+\alpha)-(x+\beta))}{\sin(x+\alpha)\sin(x+\beta)} dx$$

$$= \frac{1}{\sin(\alpha-\beta)} \int \frac{\sin(x+\alpha)\cos(x+\beta) - \sin(x+\beta)\cos(x+\alpha)}{\sin(x+\alpha)\sin(x+\beta)} dx$$

$$=\frac{1}{\sin(\alpha-\beta)}\left(\int\frac{\cos(x+\beta)}{\sin(x+\alpha)}dx-\int\frac{\cos(x+\alpha)}{\sin(x+\alpha)}dx\right)$$

$$=\frac{1}{\sin(\alpha-\beta)}\ln\left|\frac{\sin(x+\beta)}{\sin(x+\alpha)}\right|+C,\ \ \sin(\alpha-\beta)\neq 0.$$

6. Meqë $\cos a=\cos\left(\frac{x-a}{2}-\frac{x+a}{2}\right)$ atëherë kemi:

$$\int\frac{dx}{\sin x-\sin a}=\frac{1}{2\cos a}\int\frac{\cos\left(\left(\frac{x-a}{2}\right)-\left(\frac{x+a}{2}\right)\right)}{\sin\frac{x-a}{2}\cos\frac{x+a}{2}}$$

$$=\frac{1}{2\cos a}\int\frac{\cos\left(\frac{x-a}{2}\right)\cos\left(\frac{x+a}{2}\right)+\sin\left(\frac{x-a}{2}\right)\sin\left(\frac{x+a}{2}\right)}{\sin\frac{x-a}{2}\cos\frac{x+a}{2}}dx$$

$$=\frac{1}{2\cos a}\left[\int\frac{\cos\frac{x-a}{2}}{\sin\frac{x-a}{2}}dx+\int\frac{\sin\frac{x+a}{2}}{\cos\frac{x+a}{2}}dx\right]$$

$$=\frac{1}{\cos a}\ln\left|\frac{\sin\frac{x-a}{2}}{\cos\frac{x+a}{2}}\right|+C,\ \ \cos a\neq 0;\ \sin x\neq\sin a.$$

Të njehsohet integralet:

7. $\displaystyle\int\frac{dx}{\sin x-\cos x+3}.$ 8. $\displaystyle\int\frac{\sin^2 x\cdot\cos^2 x}{\sin^4 x+\cos^4 x}dx.$

9. $\displaystyle\int\frac{\sin x-3\cos x}{3\sin^2 x+5\cos^2 x}dx.$

Zgjidhja.

Zëvendësojmë: $\tan\frac{x}{2}=t;\ \frac{x}{2}=\arctan t;\ x=2\arctan t$

$$dx=\frac{2dt}{1+t^2};\ \sin x=\frac{2t}{1+t^2};\ \cos x=\frac{1-t^2}{1+t^2}.$$

d.m.th.

$$\int\frac{dx}{\sin x-\cos x+3}=\int\frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}+3}=2\int\frac{\frac{dt}{1+t^2}}{\frac{2t-1+t^2+3+3t^2}{1+t^2}}$$

$$=2\int\frac{dt}{4t^2+2t+2}=\int\frac{dt}{2t^2+t+1}.$$

8. Zëvendësojmë $\tan x=t;\ dx=\dfrac{dt}{1+t^2};\ \sin x=\dfrac{t}{\sqrt{1+t^2}};\ \cos x=\dfrac{1}{\sqrt{1+t^2}}.$

Atëherë

$$\int\frac{\sin^2 x\cdot\cos^2 x}{\sin^4 x+\cos^4 x}dx=\int\frac{\frac{t^2}{1+t^2}\cdot\frac{1}{1+t^2}}{\frac{t^4}{(1+t^2)^2}+\frac{1}{(1+t^2)^2}}\frac{dt}{1+t^2}=\int\frac{t^2}{(t^2+1)(t^4+1)}dt.$$

9. $$\int\frac{\sin x-3\cos x}{3\sin^2 x+5\cos^2 x}dx=\int\frac{\sin xdx}{3(1-\cos^2 x)+5\cos^2 x}$$

$$-3\int\frac{\cos xdx}{3\sin^2 x+5(1-\sin^2 x)}=\int\frac{\sin xdx}{3+2\cos^2 x}-3\int\frac{\cos xdx}{5-2\sin^2 x}$$

$$=-\int\frac{d(\cos x)}{3+2\cos^2 x}-3\int\frac{d(\sin x)}{5-2\sin^2 x}.$$

10. Të vërtetohet se

$$\int\frac{dx}{(a\sin x+b\cos x)^n}=\frac{A\sin x+B\cos x}{(a\sin x+b\cos x)^{n-1}}+C\int\frac{dx}{(a\sin x+b\cos x)^{n-2}}$$

ku A,B,C - konstante.

Zgjidhja.

Le të jetë $I_n = \int \frac{dx}{(a\sin x + b\cos x)^n}$. Duke përdor metodën parciale të integrimit merret:

$$I_n = \int \frac{d(-a\cos x + b\sin x)}{(a\sin x + b\cos x)^{n+1}} = \frac{-a\cos x + b\sin x}{(a\sin x + b\cos x)^{n+1}}$$

$$-(n+1)\int \frac{(a\cos x - b\sin x)^2}{(a\sin x + b\cos x)^{n+2}} = \frac{-a\cos x + b\sin x}{(a\sin x + b\cos x)^{n+1}}$$

$$-(n+1)\int \frac{(a\cos x - b\sin x)^2 + (b\cos x + a\sin x)^2}{(a\sin x + b\cos x)^{n+2}}dx.$$

Pra

$$I_n = \frac{1}{(n-1)(a^2+b^2)}\left((n-2)I_{n-2} + \frac{b\sin x - a\cos x}{(a\sin x + b\cos x)^{n-1}}\right)$$

gjë që duhej treguar.

Detyra për ushtrime

Të njehsohen integralet:

6. $\int \frac{dx}{\cos(x+a)\cos(x+b)}$.

7. $\int \frac{dx}{\sin(x+a)\cos(x+b)}$.

8. $\int \frac{dx}{\cos x - \cos a}$.

9. $\int \frac{dx}{\sin x + \sin a}$.

10. $\int \frac{dx}{\cos x + \cos a}$.

11. $\int \frac{dx}{\sin x + 2\cos x - 5}$.

12. $\int \frac{dx}{3\sin x - \cos x + 1}$.

13. $\int \frac{\sin^2 x dx}{\sin^2 x + \cos^4 x}$.

14. $\int \frac{\sin^2 x + \cos^4 x}{\cos^4 x + \sin^4 x}dx$.

15. $\int \frac{\sin x + 2\cos x}{3\sin^2 x + 7\cos^2 x}dx$.

16. $\int \frac{dx}{(2\sin x - 5\cos x)^3}$.

17. $\int \frac{3\sin x - 4\cos x}{4\sin^2 x + 5\cos^2 x}dx$.

18. $\int \frac{dx}{(\sin x + 3\cos x)^3}$. **19.** $\int \frac{\cos^{n-1}\frac{x+a}{2}}{\sin^{n-1}\frac{x-a}{2}} dx,$

20. $\int \left(\frac{\sin\frac{x-a}{2}}{\sin\frac{x+a}{2}} \right)^n dx,\ n \in N.$

21. Vërtetoni se vlen

$$\int \frac{dx}{(a+b\cos x)^n} = \frac{A\sin x}{(a+b\cos x)^{n-1}} + B\int \frac{dx}{(a+b\cos x)^{n-1}}$$

$$+C\int \frac{dx}{(a+b\cos x)^{n-2}}, (A, B, C)$$ - konstante, $n \in N,\ (|a| \neq |b|$.