II. TRANSFORMIMET E FUNKSIONEVE TRIGONOMETRIKE

2.1 FUNKSIONET TRIGONOMETRIKE TË SHUMËS DHE NDRYSHIMIT TË DY KËNDEVE

1. $\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta$.
2. $\sin(\alpha-\beta)=\sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta$.
3. $\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta$.
4. $\cos(\alpha-\beta)=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta$.

Duke shfrytëzuar formulat e shumës dhe ndryshimit të dy këndeve njehsoni vlerat e funksioneve trigonometrike të këndeve:

1. a) 15°; b) 75°; c) 165°.

2. a) $\frac{\pi}{12}$; b) $\frac{5\pi}{12}$; c) $\frac{7\pi}{12}$.

Njehsoni shprehjet:

3. $\sin 10^\circ\cdot\cos 20^\circ+\sin 20^\circ\cdot\cos 10^\circ$.

4. $\cos 102^\circ\cdot\cos 48^\circ-\sin 48^\circ\cdot\sin 102^\circ$.

5. $\cos 150^\circ\cdot\cos 5^\circ+\sin 150^\circ\cdot\sin 5^\circ$.

6. $\cos 3\alpha\cdot\cos 2\alpha-\sin 3\alpha\cdot\sin 2\alpha$.

7. $\sin 4\alpha\cdot\cos 7\alpha+\sin 7\alpha\cdot\cos 4\alpha$.

8. Thjeshtoni shprehjet:

a) $\cos\frac{7\pi}{10}\cos\frac{\pi}{5}+\sin\frac{7\pi}{10}\sin\frac{\pi}{5}$; b) $\sin\frac{\pi}{6}\cos\frac{\pi}{3}-\sin\frac{\pi}{3}\cos\frac{\pi}{6}$.

Vërtetoni identitetet:

9. $\cos(60^\circ+\beta)-\cos(300^\circ-\beta)=0$.

10. $\cos(30^\circ+\alpha)+\cos(150^\circ-\alpha)=0$.

11. $\frac{\cos 5\beta}{\sin 2\beta}-\frac{\sin 5\beta}{\cos 2\beta}=\cos 7\beta\cdot\sec 2\beta\cdot\operatorname{cosec} 2\beta$.

12. $1+\operatorname{tg}(2x)\cdot\operatorname{tg}x=\sec 2x$.

13. $\cos(3x+4y)\cdot\cos(2x+5y)+\sin(3x+4y)\cdot\sin(2x+5y)=\cos(x-y)$.

14. Njehsoni $\cos(\alpha-\beta)$ nëse $\sin\alpha=\sin\beta=\frac{5}{13}$, ku $\alpha\in\left(0,\frac{\pi}{2}\right)$, $\beta\in\left(\frac{\pi}{2},\pi\right)$.

15. Nëse α dhe β janë kënde të ngushtë ashtu që $\sin\alpha=\frac{2}{3}$ dhe $\sin\beta=\frac{3}{4}$ njehsoni $\cos(\alpha+\beta)$.

16. Njehsoni $\sin(\alpha+\beta)-\sin(\alpha-\beta)$ nëse $\sin\alpha=\frac{3}{5}$, $\sin\beta=-\frac{7}{25}$; $0<\alpha<90^\circ$, $180^\circ<\beta<270^\circ$.

17. Njehsoni:

a) $\sin(\alpha+\beta)$ dhe $\sin(\alpha-\beta)$, nëse $\cos\alpha=\frac{4}{5}$, $\sin\beta=-\frac{3}{5}$ dhe $\alpha\in\left(\frac{3\pi}{2},2\pi\right)$, $\beta\in\left(\pi,\frac{3\pi}{2}\right)$;

b) $\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}$, nëse $\sin\alpha=\frac{8}{17}$, $\cos\beta=\frac{3}{5}$ dhe $\alpha\in\left(\frac{\pi}{2},\pi\right)$, $\beta\in\left(\frac{3\pi}{2},2\pi\right)$;

c) $\dfrac{\sin\left(\dfrac{\pi}{4}+\alpha\right)}{\cos\left(\dfrac{\pi}{4}-\alpha\right)}$ nëse $\mathrm{tg}\alpha=-\dfrac{3}{4},\ \alpha\in\left(\dfrac{\pi}{2},\pi\right)$.

18. Njehsoni:

a) $\cos\left(\alpha-\dfrac{\pi}{3}\right)$, nëse $\cos\alpha=\dfrac{2}{5}$ dhe $\alpha\in(270^\circ,360^\circ)$;

b) $\sin\left(\dfrac{\pi}{6}+\alpha\right)$, nëse $\mathrm{tg}\alpha=2-\sqrt{3}$ dhe $\alpha\in\left(0,\dfrac{\pi}{2}\right)$;

c) $\sin\left(\alpha+\dfrac{\pi}{4}\right)$, nëse $\sin\alpha=m$ dhe $\alpha\in\left(0,\dfrac{\pi}{2}\right)$.

19. Njehsoni:

a) $\mathrm{tg}\left(\alpha+\dfrac{\pi}{4}\right)$, nëse $\sin\alpha=\dfrac{12}{13}$ dhe $\alpha\in\left(\dfrac{\pi}{2},\pi\right)$;

b) $\mathrm{ctg}(\alpha-\beta)$, nëse $\mathrm{tg}\alpha=\dfrac{2}{3},\ \mathrm{ctg}\beta=\dfrac{2}{5}$.

20. Nëse $\alpha+\beta=\dfrac{\pi}{4}$, vërtetoni se $(1+\mathrm{tg}\alpha)(1+\mathrm{tg}\beta)=2$.

21. Nëse $\alpha-\beta=\dfrac{\pi}{4},\ \cos\beta=\dfrac{17\sqrt{2}}{26}$ dhe $\dfrac{\pi}{4}<\alpha<\dfrac{\pi}{2}$, njehsoni $\sin\alpha$.

Theshtoni shprehjet:

22. $\dfrac{\sin(\alpha+\beta)-\sin\alpha\cdot\cos\beta}{\sin(\alpha-\beta)+\sin\beta\cdot\cos\alpha}$.

23. $\dfrac{\cos(\alpha+\beta)-\cos(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}$.

24. $\dfrac{\cos\alpha\cdot\cos\beta-\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\sin\alpha\cdot\sin\beta}$.

25. $\dfrac{\sin 35^\circ\cdot\cos 20^\circ-\cos 35^\circ\cdot\sin 20^\circ}{\cos 46^\circ\cdot\cos 29^\circ-\sin 46^\circ\cdot\sin 29^\circ}$.

26. Vërtetoni se $\mathrm{tg}(135^\circ-\varphi)=\dfrac{\mathrm{tg}\varphi-1}{\mathrm{tg}\varphi+1}$.

27. Njehsoni $\mathrm{tg}(x+y)$ nëse $\mathrm{tg}x=\dfrac{m}{1+m}$, $\mathrm{tg}y=\dfrac{1}{1+2m}$.

28. Nëse α, β janë kënde të ngushtë dhe nëse $\mathrm{tg}\alpha=\dfrac{1}{2}\wedge\ \mathrm{tg}\beta=\dfrac{1}{3}$, vërtetoni se vlen $\alpha+\beta=\dfrac{\pi}{4}$.

29. Nëse $\mathrm{tg}\alpha=\dfrac{p}{q}$ dhe $\mathrm{tg}\beta=\dfrac{q-p}{q+p}$, $\alpha,\ \beta\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$, atëherë $\alpha+\beta=\dfrac{\pi}{4}$ ose $\alpha+\beta=-\dfrac{3\pi}{4}$. Vërtetoni.

30. Njehsoni $\mathrm{tg}(\alpha+\beta)$ dhe $\mathrm{tg}(\alpha-\beta)$ nëse $\mathrm{tg}\alpha=\dfrac{3}{7}$ dhe $\mathrm{tg}\beta=\dfrac{4}{7}$.

31. Nëse $\mathrm{tg}\alpha=\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$, $\mathrm{tg}\beta=\dfrac{1}{\sqrt{2}}$, $\alpha,\ \beta\in\left(0,\dfrac{\pi}{2}\right)$, vërtetoni se $\alpha-\beta\neq\dfrac{\pi}{4}$.

32. Nëse $\alpha>0^\circ$, $\beta>0^\circ$, $\alpha+\beta=60^\circ$ dhe $\cos\alpha=\dfrac{\sqrt{2}}{4}(\sqrt{3}+1)$ njehsoni $\cos\beta$.

Vërtetoni identitetet:

33. $(\sin x+\sin y)^2+(\cos x+\cos y)^2=2+2\cos(x-y)$.

34. $\mathrm{ctg}2x+\mathrm{tg}x=\mathrm{cosec}2x$.

35.* $\sin(\alpha+\beta+\gamma)=\sin\alpha\cdot\cos\beta\cdot\cos\gamma+\cos\alpha\cdot\sin\beta\cdot\cos\gamma+\cos\alpha\cdot\cos\beta\cdot\sin\gamma$
$-\sin\alpha\cdot\sin\beta\cdot\sin\gamma$.

36.* $\cos(\alpha+\beta+\gamma)=\cos\alpha\cdot\cos\beta\cdot\cos\gamma-\sin\alpha\cdot\sin\beta\cdot\cos\gamma-\sin\alpha\cdot\cos\beta\cdot\sin\gamma$
$-\cos\alpha\cdot\sin\beta\cdot\sin\gamma$.

37.* $\mathrm{tg}(\alpha+\beta+\gamma)=\dfrac{\mathrm{tg}\alpha+\mathrm{tg}\beta+\mathrm{tg}\gamma-\mathrm{tg}\alpha\cdot\mathrm{tg}\beta\cdot\mathrm{tg}\gamma}{1-\mathrm{tg}\alpha\cdot\mathrm{tg}\beta-\mathrm{tg}\beta\cdot\mathrm{tg}\gamma-\mathrm{tg}\gamma\cdot\mathrm{tg}\alpha}$.

Vërtetoni se relacionet vijuese nuk varen nga x:

38. $\cos(\alpha + x)\cdot\cos(\alpha - x) - \sin(\alpha + x)\cdot\sin(\alpha - x)$ **.**

39. $\cos^2 x - 2\sin\alpha\cdot\cos x\cdot\sin(\alpha + x) + \sin^2(\alpha + x)$.

40. Nëse $2\text{tg}x = 3\text{tg}y$ vërtetoni se $\text{tg}y = \text{tg}(x - y)(2 + 3\text{tg}^2 y)$.

Vërtetoni identitetet:

41. $\sin 25^\circ\cdot\cos 35^\circ + \cos 25^\circ\cdot\sin 35^\circ = \dfrac{\sqrt{3}}{2}$.

42. $\text{tg}3\alpha - \text{tg}2\alpha - \text{tg}\alpha = \text{tg}3\alpha\cdot\text{tg}2\alpha\cdot\text{tg}\alpha$

43. $\cos x + \cos(120^\circ + x) + \cos(240^\circ + x) + \sin x = \sin x$.

44. $\dfrac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} = \dfrac{\text{tg}\alpha + \text{tg}\beta}{1 + \text{tg}\alpha\cdot\text{tg}\beta}$.

45. $\text{tg}\alpha + \text{tg}\beta = \dfrac{2\sin(\alpha+\beta)}{\cos(\alpha+\beta) + \cos(\alpha-\beta)} = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cdot\cos\beta}$.

46. $\dfrac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \text{ctg}\left(\dfrac{\pi}{4} + \alpha\right)$.

47. $\text{tg}(\alpha+\beta) - \text{tg}\alpha - \text{tg}\beta = \text{tg}\alpha\cdot\text{tg}\alpha\cdot\text{tg}(\alpha+\beta)$.

48. $\cos(\alpha+\beta) = \dfrac{1 - \text{tg}\alpha\cdot\text{tg}\beta}{1 + \text{tg}\alpha\cdot\text{tg}\beta}\cdot\cos(\alpha-\beta) = \sin\alpha\cdot\sin\beta(\text{ctg}\alpha\cdot\text{ctg}\beta - 1)$.

49.* Vërtetoni se $\text{tg}\alpha + \text{tg}\beta + \text{tg}\gamma = \text{tg}\alpha\cdot\text{tg}\beta\cdot\text{tg}\gamma$ nëse $\alpha + \beta + \gamma = \pi$.

50.* Vërtetoni se $\alpha + \beta + \gamma = \pi$, nëse α, β, dhe γ janë kënde të ngushtë dhe

$$\text{ctg}\alpha\cdot\text{ctg}\beta + \text{ctg}\alpha\cdot\text{ctg}\gamma + \text{ctg}\beta\cdot\text{ctg}\gamma = 1 .$$

51.* Nëse për këndet α, β dhe γ të trekëndëshit vlen barazimi $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$, vërtetoni se trekëndëshi është kënddrejtë.

52.* Vërtetoni se nëse $\operatorname{tg}\alpha=\frac{1}{12}$, $\operatorname{tg}\beta=\frac{2}{5}$, $\operatorname{tg}\gamma=\frac{1}{3}$, ku $\alpha,\ \beta,\ \gamma\in\left(0,\frac{\pi}{2}\right)$, atëherë vlen $\alpha+\beta+\gamma=\frac{\pi}{4}$.

53.* Vërtetoni se nëse $\sin x+\sin y=a$ dhe $\cos x+\cos y=b$ atëherë

$$\cos(x-y)=\frac{a^2+b^2-2}{2}.$$

54.* Tregoni se është i vërtetë implikacioni

$$\frac{\operatorname{tg}(x-y)}{\operatorname{tg}x}+\frac{\sin^2 z}{\sin^2 x}=1\Rightarrow \operatorname{tg}^2 z=\operatorname{tg}x\cdot\operatorname{tg}y.$$

55.* Vërtetoni identitetin

$$\frac{\sqrt{2}\cdot\cos\alpha-2\cdot\cos\left(\frac{\pi}{4}+\alpha\right)}{2\sin\left(\frac{\pi}{4}+\alpha\right)-\sqrt{2}\sin\alpha}=\operatorname{tg}\alpha.$$

56.* Tregoni se është i vërtetë implikacioni

$$A\cdot\sin(\alpha+\beta)=\sin\alpha\Rightarrow \operatorname{tg}(\alpha+\beta)=\frac{\sin\beta}{\cos\beta-A}.$$

57.* Tregoni se nëse për α,β vlen $\frac{\sin\beta}{\sin(2\alpha+\beta)}=\frac{n}{m}$ $(|m|>|n|)$ atëherë

$$\frac{1+\frac{\operatorname{tg}\beta}{\operatorname{tg}\alpha}}{m+n}=\frac{1-\operatorname{tg}\alpha\cdot\operatorname{tg}\beta}{m-n}.$$

2.2 FUNKSIONET TRIGONOMETRIKE TË KËNDIT TË DYFISHTË

1. $\sin 2\alpha = 2\sin\alpha\cdot\cos\alpha$.
2. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$.
3. $\mathrm{tg}2\alpha = \dfrac{2\mathrm{tg}\alpha}{1-\mathrm{tg}^2\alpha},\ \alpha \neq \dfrac{\pi}{2}(2k+1),\ \alpha \neq \dfrac{\pi}{4}(2n+1)$, ku $n \in Z$.
4. $\mathrm{ctg}2\alpha = \dfrac{\mathrm{ctg}^2\alpha - 1}{2\mathrm{ctg}\alpha},\ \alpha \neq \dfrac{k\pi}{2},\ k \in Z$.

Vërtetoni se:

1. a) $2\cos^2\alpha - \cos 2\alpha = 1$; b) $\cos^2 2\alpha - 4\sin^2\alpha\cdot\cos^2\alpha = \cos 4\alpha$;

c) $2\sin^2\alpha + \cos 2\alpha = 1$.

2. a) $\sin 3\alpha = 3\cdot\sin\alpha - 4\cdot\sin^3\alpha$; b) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$;

c) $\mathrm{tg}3\alpha = \dfrac{3\mathrm{tg}\alpha - \mathrm{tg}^3\alpha}{1-3\mathrm{tg}^2\alpha},\ \alpha \neq \dfrac{k\pi}{3} + \dfrac{\pi}{2},\ k \in Z$;

d) $\mathrm{ctg}3\alpha = \dfrac{-3\mathrm{ctg}\alpha + \mathrm{ctg}^3\alpha}{3\mathrm{ctg}^2\alpha - 1},\ \alpha \neq \dfrac{k\pi}{3},\ k \in Z$.

3. a) $\mathrm{tg}4\alpha = \dfrac{4\mathrm{tg}\alpha - 4\mathrm{tg}^3\alpha}{1-6\mathrm{tg}^2\alpha + \mathrm{tg}^4\alpha}$; b) $\mathrm{ctg}4\alpha = \dfrac{\mathrm{ctg}^4\alpha - 6\mathrm{ctg}^2\alpha + 1}{4\mathrm{ctg}^3\alpha - 4\mathrm{ctg}\alpha}$.

4. Nëse $\mathrm{tg}2x = 1$, njehsoni $\mathrm{tg}\left(\dfrac{5\pi}{4} + x\right) - \mathrm{tg}\left(\dfrac{5\pi}{4} - x\right)$.

5. Të njehsohen sin2α, cos2α, tg2α dhe ctg2α nëse:

a) $\sin\alpha = -0{,}6$ dhe $\pi < \alpha < \dfrac{3\pi}{2}$; b) $\sin\alpha = \dfrac{3}{5}$ dhe $\dfrac{\pi}{2} < \alpha < \pi$;

c) $\cos\alpha = \dfrac{4}{5}$ dhe $\dfrac{3\pi}{2} < \alpha < 2\pi$.

6. Njehsoni tg2*x*, nëse $\sin x = \dfrac{m-n}{m+n}$, për $m \neq -n$ dhe $(m-n) \neq \pm 2mn$.

7. Nëse tgα= $\sqrt{3}+2$ ku $\alpha \in \left(0, \frac{\pi}{2}\right)$, njehsoni sin2α.

8. Nëse $\frac{1+\sin^4\alpha+\cos^4\alpha}{\sin^2\cdot\cos^2}=7$ vërtetoni se $\sin^2 2\alpha=\frac{8}{9}$.

9. Vërtetoni se $\sin 15^\circ\cdot\cos 15^\circ-4\sin^2\alpha\cdot\cos^2\alpha=\cos^2 2\alpha-\frac{3}{4}$.

10.* Thjeshtoni shprehjen $A=\cos\frac{\pi}{7}\cdot\cos\frac{4\pi}{7}\cdot\cos\frac{5\pi}{7}$.

11.* Krahasoni:

a) tg2α dhe 2tgα, nëse $\left(0<\alpha<\frac{\pi}{4}\right)$;

b) sin2α dhe 2sinα, nëse $\left(0<\alpha<\frac{\pi}{2}\right)$.

Vërtetoni identitetet:

12. $2\sin^2\alpha+\frac{1+\cos 2\alpha}{1-\cos 2\alpha}+\cos 2\alpha=\frac{1}{\sin^2\alpha}$.

13. $\frac{1-2\sin^2\alpha}{\sin\alpha\cdot\cos\alpha}=2\mathrm{ctg}2\alpha$.

14. $\frac{\sin 4\alpha}{2\sin 2\alpha}=\cos^2\alpha-\sin^2\alpha$.

15. $\cos^4\alpha+\sin^4\alpha=1-0.5\cdot\sin^2 2\alpha$.

16. $\cos^6\alpha+\sin^6\alpha=1-0.75\cdot\sin^2 2\alpha$.

17. $\frac{\cos\alpha}{\cos\alpha-\sin\alpha}-\frac{\sin\alpha}{\cos\alpha+\sin\alpha}=\sec 2\alpha$.

18. $\frac{\cos\beta}{\cos\beta-\sin\beta}+\frac{\sin\beta}{\cos\beta+\sin\beta}=1+\mathrm{tg}2\beta$.

19. $\frac{\cos 2\alpha}{-\sin\alpha+\cos\alpha}=\frac{1+\sin 2\alpha}{\sin\alpha+\cos\alpha}$.

20. $\frac{1-\mathrm{tg}\gamma}{1+\mathrm{tg}\gamma}=\frac{1-\sin 2\gamma}{\cos 2\gamma}$.

21. $\frac{1-\mathrm{tg}^2\theta}{1+\mathrm{tg}^2\theta}=\cos 2\theta$.

22. $\frac{1+\sin 2\alpha+\cos 2\alpha}{1+\sin 2\alpha-\cos 2\alpha}=\mathrm{ctg}\alpha$.

23. $\frac{\sin\beta+\sin 2\beta}{1+\cos\beta+\cos 2\beta}=\mathrm{tg}\beta$.

24. $\frac{\sin 2\alpha-\sin\alpha}{1-\cos\alpha-\cos 2\alpha}=\mathrm{tg}\alpha=\frac{1-\cos 2\alpha+\sin 2\alpha}{1+\cos 2\alpha+\sin 2\alpha}$.

25. $\frac{2-\sin 4\alpha\cdot\mathrm{tg}2\alpha}{\sin 4\alpha}=\mathrm{ctg}2\alpha$.

26. $\frac{(1-\sin^2 2\alpha)((1-\mathrm{tg}^2\alpha)^2+4\mathrm{tg}^2\alpha)}{(1+\sin^2 2\alpha)((1+\mathrm{tg}^2\alpha)^2-4\mathrm{tg}^2\alpha)}=\frac{1}{1+\sin^2 2\alpha}$.

27. * Nëse tgα= $\mathrm{tg}\alpha=\frac{1}{7}$, $\sin\beta=\frac{1}{\sqrt{10}}$, $0<\alpha<\frac{\pi}{2}$ dhe $0<\beta<\frac{\pi}{2}$ atëherë $\alpha+2\beta=\frac{\pi}{4}$. Vërtetoni.

28.* Vërtetoni se nëse $\sin\alpha=\frac{1}{\sqrt{5}}$, $\sin\beta=\frac{1}{\sqrt{10}}$ dhe $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$, atëherë $\alpha+\beta=\frac{\pi}{4}$.

29. Njehsoni $\mathrm{tg}3\alpha$ nëse $\mathrm{tg}\alpha=\frac{4}{3}$.

30. Nëse $\cos\alpha=\frac{5}{13}$ dhe $0<\alpha<\frac{\pi}{2}$, njehsoni:

a) $\sin 2\alpha$; b) $\cos 2\alpha$; c) $\mathrm{tg}2\alpha$; d) $\mathrm{ctg}2\alpha$.

Duke u nisur nga fakti se $\sin 2\cdot 18^\circ=\cos 3\cdot 18^\circ$, njehsoni:

31. a) $\sin 18^\circ$; b) $\cos 18^\circ$; c) $\mathrm{tg}18^\circ$; d) $\mathrm{ctg}18^\circ$.

32. a) $\sin 36^\circ$; b) $\cos 36^\circ$; c) $\mathrm{tg}36^\circ$; d) $\mathrm{ctg}36^\circ$.

Vërtetoni identitetet:

33.* $\cos\alpha\cdot\cos 2\alpha\cdot\cos 4\alpha\cdot\cos 8\alpha=\frac{\sin 16\alpha}{16\sin\alpha}$.

34.* $\cos\frac{\pi}{65}\cdot\cos\frac{2\pi}{65}\cdot\cos\frac{4\pi}{65}\cdot\cos\frac{8\pi}{65}\cdot\cos\frac{16\pi}{65}\cdot\cos\frac{32\pi}{65}=\frac{1}{64}$.

35.* $\mathrm{tg}^2 36^\circ\cdot\mathrm{tg}^2 72^\circ=5$.

36.* $\mathrm{tg}x\cdot\mathrm{tg}\left(\frac{\pi}{3}-x\right)\cdot\mathrm{tg}\left(\frac{\pi}{3}+x\right)=\mathrm{tg}3x$.

37.* $\sin x\cdot\sin\left(\frac{\pi}{3}-x\right)\cdot\sin\left(\frac{\pi}{3}+x\right)=\frac{1}{4}\sin 3x$.

38.* $\cos x\cdot\cos\left(\frac{\pi}{3}-x\right)\cdot\cos\left(\frac{\pi}{3}+x\right)=\frac{1}{4}\cos 3x$.

39.* $\sin 20^\circ\cdot\sin 40^\circ\cdot\sin 60^\circ\cdot\sin 80^\circ=\frac{3}{16}$.

40.* $\cos 10^\circ\cdot\cos 30^\circ\cdot\cos 50^\circ\cdot\cos 70^\circ=\frac{3}{16}$.

41.* $\frac{\mathrm{tg}6^\circ\cdot\mathrm{tg}54^\circ\cdot\mathrm{tg}66^\circ}{\mathrm{tg}18^\circ}=\mathrm{tg}45^\circ$.

2.3 FUNKSIONET TRIGONOMETRIKE TË GJYSMËKËNDIT

1. $\left|\sin\frac{\alpha}{2}\right|=\sqrt{\frac{1-\cos\alpha}{2}}$.

2. $\left|\cos\frac{\alpha}{2}\right|=\sqrt{\frac{1+\cos\alpha}{2}}$.

3. $\left|\mathrm{tg}\frac{\alpha}{2}\right|=\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$, për $\alpha\neq\pi(2k+1),\ k\in Z$.

4. $\left|\mathrm{ctg}\frac{\alpha}{2}\right|=\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$, për $\alpha\neq 2k\pi,\ k\in Z$.

1. Të vërtetohen formulat e mëposhtme:

a) $\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}$; b) $\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$.

2. Të njehsohen vlerat e funksioneve trigonometrike për këndet:

a) $\alpha=\frac{\pi}{8}$; b) $\alpha=\frac{\pi}{12}$; c) $\alpha=\frac{\pi}{16}$; d) $\alpha=\frac{\pi}{24}$.

3. Njehsoni $\sin\frac{\alpha}{2}$, $\cos\frac{\alpha}{2}$, dhe $\mathrm{tg}\frac{\alpha}{2}$, nëse:

a) $\cos\alpha=-\frac{7}{25}$, $\alpha\in\left(\frac{\pi}{2},\pi\right)$; b) $\sin\alpha=-\frac{15}{17}$, $\alpha\in\left(\frac{3\pi}{2},2\pi\right)$.

4. Nëse α është kënd i ngushtë dhe $\cos\alpha=\frac{3}{5}$, njehsoni $\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}$.

Vërtetoni se:

5. a) $\sin\alpha=\dfrac{2\mathrm{tg}\frac{\alpha}{2}}{1+\mathrm{tg}^2\frac{\alpha}{2}}$, për $\alpha\neq\pi(2k+1)$, $k\in Z$;

b) $\cos\alpha=\dfrac{1-\mathrm{tg}^2\frac{\alpha}{2}}{1+\mathrm{tg}^2\frac{\alpha}{2}}$, për $\alpha\neq\pi(2k+1)$, $k\in Z$;

c) $\mathrm{tg}\alpha=\dfrac{2\mathrm{tg}\frac{\alpha}{2}}{1-\mathrm{tg}^2\frac{\alpha}{2}}$, për $\alpha\neq\pi(2k+1)$ dhe $\alpha\neq\frac{\pi}{2}(2n+1)$, $k, n\in Z$;

d) $\mathrm{ctg}\alpha=\dfrac{1-\mathrm{tg}^2\frac{\alpha}{2}}{2\mathrm{tg}\frac{\alpha}{2}}$, për $\alpha\neq k\pi$, $k\in Z$.

6. a) $\mathrm{tg}\frac{\alpha}{2}=\dfrac{1-\cos\alpha}{\sin\alpha}$, $\alpha\neq k\pi$, $k\in Z$;

b) $\mathrm{tg}\frac{\alpha}{2}=\dfrac{\sin\alpha}{1+\cos\alpha}$, $\alpha\neq\pi(2k+1)$, $k\in Z$.

Thjeshtoni shprehjet:

7. a) $\frac{\sin\alpha}{1-\cos\alpha}$; b) $\frac{1-\cos 2\alpha}{\sin 2\alpha}$; c) $\frac{1-\cos\alpha}{1+\cos\alpha}$.

8. a) $1-\cos^2\left(\frac{\pi}{4}-\frac{4x}{3}\right)$; b) $2\cos^2\left(\frac{\pi}{4}+\frac{3x}{2}\right)-1$.

9. $1-2\sin^2\left(\frac{\pi}{4}-\frac{5x}{2}\right)+2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)$.

10. $4\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)$.

11. Nëse $f\left(\mathrm{tg}\frac{\alpha}{2}\right)=\cos\alpha$ njehsoni $f(\cos\alpha)$.

12. Njehsoni $\mathrm{tg}\frac{\alpha}{2}$ nëse $\cos 2\alpha=\frac{7}{32}$ dhe $\alpha\in\left(-\pi,-\frac{3\pi}{4}\right)$.

13. Njehsoni $\sin\frac{\alpha}{4}$ nëse $\sin\alpha=\frac{336}{625}$ dhe $\alpha\in(450^\circ,540^\circ)$.

14. Njehsoni: $\sin k\frac{\alpha}{2}$, $\cos k\frac{\alpha}{2}$, $\mathrm{tg}k\frac{\alpha}{2}$ nëse:

a) $\cos\alpha=0{,}8$ dhe $\alpha\in\left(0,\frac{\pi}{2}\right)$;

b) $\mathrm{tg}\alpha=\frac{24}{7}$, $\alpha\in(180^\circ,270^\circ)$ ku $k\in\{1,2,3,4\}$.

15. Nëse $\mathrm{tg}\frac{\alpha}{2}+\mathrm{ctg}\frac{\alpha}{2}=m$ njehsoni:

a) $\mathrm{tg}^2\frac{\alpha}{2}+\mathrm{ctg}^2\frac{\alpha}{2}$; b) $\mathrm{tg}^3\frac{\alpha}{2}+\mathrm{ctg}^3\frac{\alpha}{2}$; c) $\mathrm{tg}\frac{\alpha}{2}-\mathrm{ctg}\frac{\alpha}{2}$.

16. Nëse $\mathrm{tg}\frac{x}{2}=a$ tregoni se

$$\sin x=\frac{2a}{1+a^2},\ \cos x=\frac{1-a^2}{1+a^2},\ \mathrm{tg}x=\frac{2a}{1-a^2},\ \mathrm{ctg}x=\frac{1-a^2}{2a}.$$

17.* Nëse $n>1,\ x$ është kënd i ngushtë i tillë që $\left(\frac{\pi}{4}<x<\frac{\pi}{2}\right)$ dhe $\cos x=\frac{2n}{1+n^2}$, vërtetoni se $\sin\frac{x}{2}\cdot\cos^3\frac{x}{2}=\frac{1}{4}\frac{(n^2-1)(n+1)^2}{(n^2+1)^2}$. Çfarë do të merret sikur $\left(0<x<\frac{\pi}{4}\right)$?

18.* Nëse $\sin x-\cos x=m$ dhe $|m|\le\sqrt{2}$, $m\ne-1$, njehsoni $\mathrm{tg}\frac{x}{2}$.

19. Nëse $\sin\alpha+\cos\alpha=m$, njehsoni vlerën e shprehjes $\dfrac{1+\cos 2\alpha}{\mathrm{ctg}\frac{\alpha}{2}-\mathrm{tg}\frac{\alpha}{2}}$.

20. Nëse $\mathrm{tg}\frac{\alpha}{2}=x$, vërtetoni se $\dfrac{1-2\sin^2\frac{\alpha}{2}}{1+\sin\alpha}=\dfrac{1-x}{1+x}$.

Njehsoni:

21. $\sqrt{1+\sin\alpha}+\sqrt{1-\sin\alpha}$, nëse $a^2=\sin\frac{\alpha}{2}$, $\frac{\pi}{2}<\alpha<\pi$.

22. $\sqrt{1+\cos\alpha}+\sqrt{1-\cos\alpha}$, nëse $a^2=\cos\frac{\alpha}{2}$, $\frac{\pi}{2}<\alpha<\pi$.

23. Njehsoni $16\cdot\sin\frac{\alpha}{2}\cdot\sin\frac{3\alpha}{2}$ nëse $\cos\alpha=\frac{3}{4}$, $0<\frac{\alpha}{2}<\frac{\pi}{2}$.

Thjeshtoni shprehjet:

24. a) $\dfrac{1+\cos\alpha}{1-\cos\alpha}\cdot\mathrm{tg}^2\frac{\alpha}{2}-\cos^2\alpha$; b) $\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}\cdot\mathrm{ctg}\frac{\alpha}{2}-\sin^2\alpha$.

25. a) $\dfrac{1-\sin\alpha}{1+\sin\alpha}-\operatorname{tg}^2\left(\dfrac{\pi}{4}-\dfrac{\alpha}{2}\right)$; b) $1+\operatorname{tg}^2\left(\dfrac{\pi}{4}-\dfrac{\alpha}{2}\right)$.

Vërtetoni identitetet:

26. $\dfrac{\operatorname{tg}\frac{\alpha}{2}+\operatorname{ctg}\frac{\alpha}{2}}{\operatorname{ctg}\frac{\alpha}{2}-\operatorname{tg}\frac{\alpha}{2}}=\sec\alpha$.

27. $\operatorname{tg}(45^\circ-\alpha)^2=\dfrac{(\sin\alpha-\cos\alpha)^2}{(\sin\alpha+\cos\alpha)^2}$.

28. $\dfrac{2}{\operatorname{ctg}\frac{\alpha}{2}+\operatorname{tg}\frac{\alpha}{2}}=\sin\alpha$.

29. $\operatorname{tg}\dfrac{\alpha}{2}=\dfrac{\sin\alpha}{1+\cos\alpha}$.

30.* $\sqrt{\cos x+\operatorname{ctg}x}+\sqrt{\operatorname{ctg}x-\cos x}=2\cos\dfrac{x}{2}\sqrt{\operatorname{ctg}x},\ 0<x\le\dfrac{\pi}{2}$.

31.* $\log(\cos^2(\alpha+\beta)+\cos^2(\alpha-\beta)-\cos 2\alpha\cdot\cos 2\beta)=0$.

Vërtetoni se:

32.* $\sin\alpha+\sin\beta+\sin\gamma=4\cos\dfrac{\alpha}{2}\cdot\cos\dfrac{\beta}{2}\cdot\cos\dfrac{\gamma}{2}$, nëse $\alpha+\beta+\gamma=\pi$.

33.* $\cos\alpha+\cos\beta+\cos\gamma=1+4\sin\dfrac{\alpha}{2}\cdot\sin\dfrac{\beta}{2}\cdot\sin\dfrac{\gamma}{2}$, nëse $\alpha+\beta+\gamma=\pi$.

34.* Nëse $\cos\alpha=\dfrac{a}{b+c}$, $\cos\beta=\dfrac{b}{a+c}$, $\cos\gamma=\dfrac{c}{a+b}$, atëherë

$$\operatorname{tg}^2\frac{\alpha}{2}+\operatorname{tg}^2\frac{\beta}{2}+\operatorname{tg}^2\frac{\gamma}{2}=1.$$

35.* Tregoni se shprehja $\sqrt{4\cos^4 x-6\cos 2x+3}+\sqrt{4\sin^4 x+6\cos 2x+3}$, nuk varet nga x-i.

36.* Vërtetoni se nëse $\sin x$ dhe $\cos x$ janë numra racional atëherë dhe vetëm atëherë nëse $\operatorname{tg}\dfrac{x}{2}$ është numër racional ose nuk ekziston fare.

2.4 TRANSFORMIMI I PRODHIMIT TË FUNKSIONEVE TRIGONOMETRIKE NË SHUMË DHE NDRYSHIM

1. $\sin\alpha\cdot\sin\beta=\frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$.

2. $\cos\alpha\cdot\cos\beta=\frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta))$.

3. $\sin\alpha\cdot\cos\beta=\frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))$.

1. Transformoni shprehjet e dhëna:

a) $\sin^2\alpha$; b) $\cos^2\alpha$; c) $\sin^3\alpha$; d) $\cos^3\alpha$;
e) $\sin^4\alpha$; f) $\cos^4\alpha$; g) $\mathrm{tg}^3\alpha$.

2. Njehsoni

a) $3\sin\frac{2\pi}{3}\cdot\cos\frac{5\pi}{6}$; b) $3\sin\left(2+\frac{\pi}{3}\right)\cdot\cos\left(2-\frac{\pi}{6}\right)$.

Transformoni prodhimet e dhëna:

3. a) $\cos 4\alpha\cdot\cos 5\alpha$; b) $\sin 3x\cdot\sin 5x$;
c) $\sin(\alpha-\beta)\cdot\cos(\alpha+\beta)$.

4. a) $\sin\left(\frac{\pi}{4}-x\right)\cdot\cos\left(\frac{\pi}{4}+x\right)$; b) $2\sin 65^\circ\cdot\cos 20^\circ$;
c) $\cos(x+25^\circ)\cdot\cos(y+20^\circ)$.

5. a) $\sin x\cdot\sin 2x\cdot\sin 4x$; b) $\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}$;
c) $\sin x\cdot\cos 3x\cdot\sin 4x$.

Vërtetoni identitetet:

6. $\sin(45^\circ-\alpha)\cdot\sin(45^\circ+\alpha)\cdot\cos 2\alpha=\frac{1+\cos 4\alpha}{4}$.

7. $\cos^2 3 + \cos^2 1 - \cos 2 \cdot \cos 4 = 1$.

8. $\sin^2 \alpha \cdot \cos \alpha = \frac{1}{4}(\cos \alpha - \cos 3\alpha)$.

9. $\sin^2 \alpha \cos^3 \alpha = \frac{1}{16}(2\cos \alpha - \cos 3\alpha - \cos 5\alpha)$.

10. $\sin^3 \alpha \cdot \cos^3 \alpha = \frac{1}{32}(3\sin 2\alpha - \sin 6\alpha)$.

11.* Vërtetoni se nëse për këndet e trekëndëshit vlen relacioni

$$\sin \alpha = 2\sin\frac{\beta}{2} \cdot \left(\sin\frac{\alpha+\beta}{2} + \sin\frac{\alpha-\beta}{2}\right)$$

atëherë ai trekëndësh është barakrahës.

12.* Duke zbatuar formulat për transformimin e prodhimit në shumë dhe ndryshim vërtetoni se shprehja

$$\frac{\sin(x+y)\cdot\cos(x-y)+\cos(x+y)\cdot\sin(x-y)}{\cos(x+y)\cdot\cos(x-y)-\sin(x+y)\cdot\sin(x-y)}$$

nuk varet nga y.

Duke zbatuar formulat për transformimin e prodhimit në shumë dhe ndryshim njehsoni vlerën numerike të shprehjeve vijuese:

13.* a) $\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 60^\circ \cdot \cos 80^\circ$;

b) $\text{ctg} 20^\circ \cdot \text{ctg} 40^\circ \cdot \text{ctg} 60^\circ \cdot \text{ctg} 80^\circ$.

14.* $\sin^2\frac{\pi}{7} \cdot \sin^2\frac{2\pi}{7} \cdot \sin^2\frac{3\pi}{7}$.

15.* $\cos\frac{\pi}{7} \cdot \cos\frac{2\pi}{7} \cdot \cos\frac{4\pi}{7}$.

16.* $\text{tg}\frac{\pi}{7} \cdot \text{tg}\frac{2\pi}{7} \cdot \text{tg}\frac{3\pi}{7}$.

Vërtetoni se:

17.* $\sin 10^\circ \cdot \sin 30^\circ \cdot \sin 50^\circ \cdot \sin 70^\circ = \frac{1}{16}$.

18.* $\sin^2 10^\circ \cdot \sin^2 50^\circ \cdot \sin^2 70^\circ = \dfrac{1}{64}$.

19.* $\operatorname{tg} 20^\circ \cdot \operatorname{tg} 40^\circ \cdot \operatorname{tg} 60^\circ \cdot \operatorname{tg} 80^\circ = 3$.

Nëse $\alpha + \beta + \gamma = \pi$, vërtetoni se:

20. $\sin\alpha \cdot \sin\beta \cdot \sin\gamma = \dfrac{1}{4}(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)$.

21. $\sin\alpha \cdot \cos\beta \cdot \cos\gamma = \dfrac{1}{4}(\sin 2\gamma - \sin 2\alpha + \sin 2\beta)$.

22. $\sin\alpha \cdot \sin\beta \cdot \cos\gamma = \dfrac{1}{4}(\cos 2\gamma - \cos 2\alpha - \cos 2\beta + 1)$.

23. $\cos\alpha \cdot \cos\beta \cdot \cos\gamma = \dfrac{1}{4}(-\cos 2\alpha - \cos 2\beta - \cos 2\gamma - 1)$.

24.* $\operatorname{tg}\dfrac{\alpha}{2} \cdot \operatorname{tg}\dfrac{\beta}{2} + \operatorname{tg}\dfrac{\beta}{2} \cdot \operatorname{tg}\dfrac{\gamma}{2} + \operatorname{tg}\dfrac{\gamma}{2} \cdot \operatorname{tg}\dfrac{\alpha}{2} = 1$.

25.* $\sin\alpha \cdot \cos\beta \cdot \cos\gamma + \sin\beta \cdot \cos\alpha \cdot \cos\gamma + \sin\gamma \cdot \cos\alpha \cdot \cos\beta = \sin\alpha \cdot \sin\beta \cdot \sin\gamma$.

26.* $\sin\dfrac{\alpha}{2} \cdot \sin\dfrac{\beta}{2} \cdot \cos\dfrac{\gamma}{2} + \sin\dfrac{\alpha}{2} \cdot \cos\dfrac{\beta}{2} \cdot \sin\dfrac{\gamma}{2} + \cos\dfrac{\alpha}{2} \cdot \sin\dfrac{\beta}{2} \cdot \sin\dfrac{\gamma}{2}$

$= \cos\dfrac{\alpha}{2} \cdot \cos\dfrac{\beta}{2} \cdot \cos\dfrac{\gamma}{2}$.

27.* $\operatorname{tg}2\alpha \cdot \operatorname{tg}(30^\circ - \alpha) + \operatorname{tg}2\alpha \cdot \operatorname{tg}(60^\circ - \alpha) + \operatorname{tg}(60^\circ - \alpha) \cdot \operatorname{tg}(30^\circ - \alpha) = 1$.

28.* Thjeshtoni shprehjen

$A = \cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ ... \ + \cos(\alpha + n \cdot \beta),\ n \in N$.

29.* Nëse $\cos(x + y) = \dfrac{1}{3}$ dhe $\cos(x - y) = \dfrac{1}{2}$, ku $0 < x < y < 90^\circ$, njehsoni $\cos x$ dhe $\cos y$.

30.* Le të jetë $\cos(\alpha+\beta)=x$ dhe $\cos(\alpha-\beta)=y$, ku $0<x<y<90^\circ$. Nëse $0<\alpha<\beta<90^\circ$ të shprehen $\sin\alpha$, $\sin\beta$, $\cos\alpha$ dhe $\cos\beta$ në varësi të *x*-it dhe *y*-it.

2.5 TRANSFORMIMI I SHUMËS DHE NDRYSHIMIT TË FUNKSIONEVE TRIGONOMETRIKE NË PRODHIM

1. $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$.

2. $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$.

3. $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$.

4. $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$.

Të transformohen në prodhim:

1. a) $\sin 9a + \sin 3a$; b) $\sin 3a - \sin 4a$;

c) $\cos a + \cos 6a$; d) $\cos 3b - \cos\frac{b}{2}$.

2. a) $\sin(3x+a) + \sin(3x-a)$; b) $\cos\left(\frac{x}{2}+a\right) - \cos\left(a-\frac{x}{2}\right)$;

c) $\cos(x+2y+z) + \cos(x+4y-z)$.

3. a) $\sin 20^\circ + \sin 70^\circ$; b) $\cos 101^\circ - \cos 11^\circ$;

c) $\frac{\sqrt{3}}{2} - \sin 30^\circ$; d) $\sin\alpha + 1$.

4. a) $\sin a - \cos a$; b) $\cos\frac{\pi}{5} - \sin\frac{2\pi}{5}$; c) $2\cos\alpha + 1$;

d) $\sqrt{3} + 2\cos x$; e) $1 - 4\cos^2\alpha$; f) $3 - 4\sin^2\alpha$.

Vërtetoni identitetet:

5. $\dfrac{\cos a-\cos 3a}{\sin 3a-\sin a}=\operatorname{tg}2a.$

6. $\dfrac{\sin\left(\dfrac{\pi}{4}+\alpha\right)+\sin\left(\dfrac{\pi}{4}-\alpha\right)}{\sin\left(\dfrac{\pi}{6}+\alpha\right)+\sin\left(\dfrac{\pi}{6}-\alpha\right)}=\sqrt{2}.$

7. $\dfrac{\cos(2a-3b)+\cos 3b}{\sin 3b+\sin(2a-3b)}=\operatorname{tg}(3b-a).$

8. $\sin^2\left(\dfrac{\pi}{8}+a\right)-\sin^2\left(\dfrac{\pi}{8}-a\right)=\dfrac{\sin 2\alpha}{\sqrt{2}}.$

9. $\dfrac{\cos(x-3y)-\cos(3x+y)}{\sin(3x+y)+\sin(x-3y)}=\operatorname{tg}(x+2y).$

10. $\dfrac{\sin a+\sin 2a+\sin 3a}{\cos a+\cos 2a+\cos 3a}=\operatorname{tg}2a.$

11. $\dfrac{2\cos(10^\circ+2\alpha)-1}{2\cos(10^\circ+2\alpha)+1}=\operatorname{tg}(35^\circ+\alpha)\cdot\operatorname{tg}(25^\circ-\alpha).$

12. $\dfrac{\sin\alpha\pm\sin 3\alpha+\sin 5\alpha}{\cos\alpha\pm\cos 3\alpha+\cos 5\alpha}=\operatorname{tg}3\alpha.$

13. Tregoni se nëse $\sin\gamma=\dfrac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}$ atëherë trekëndëshi është kënddrejtë.

Vërtetoni se:

14.* $\sec 7^\circ(\cos 43^\circ+\cos 29^\circ-\cos 79^\circ-\cos 75^\circ)=1.$

15.* $\operatorname{tg}9^\circ-\operatorname{tg}27^\circ-\operatorname{tg}63^\circ+\operatorname{tg}81^\circ=4.$

16.* $\dfrac{1}{\sin 10^\circ}-\dfrac{\sqrt{3}}{\cos 10^\circ}=4.$

17.* $4(\cos^3 20^\circ+\cos^3 40^\circ)=3\sqrt{3}\cos 10^\circ.$

18.* $\operatorname{tg}30^\circ + \operatorname{tg}40^\circ + \operatorname{tg}50^\circ + \operatorname{tg}60^\circ = \dfrac{8\sqrt{3}\cos 20^\circ}{3}.$

19.* $\operatorname{tg}^6 20^\circ - 33\cdot\operatorname{tg}^4 20^\circ + 27\cdot\operatorname{tg}^2 20^\circ = 3.$

20.* $\sin a - \sin 2a + \sin 3\alpha = 4\sin\dfrac{a}{2}\cdot\cos a\cdot\cos\dfrac{3a}{2}.$

Nëse $\alpha+\beta+\gamma=\pi$, tregoni se vlejnë identitetet:

21.* $\sin\alpha+\sin\beta+\sin\gamma = 4\cos\dfrac{\alpha}{2}\cdot\cos\dfrac{\beta}{2}\cdot\cos\dfrac{\gamma}{2}.$

22.* $\sin\alpha+\sin\beta-\sin\gamma = 4\sin\dfrac{\alpha}{2}\cdot\sin\dfrac{\beta}{2}\cdot\cos\dfrac{\gamma}{2}.$

23.* $\sin 2\alpha+\sin 2\beta+\sin 2\gamma = 4\sin\alpha\cdot\sin\beta\cdot\sin\gamma.$

24.* $\sin 2\alpha-\sin 2\beta-\sin 2\gamma = -4\sin\alpha\cdot\cos\beta\cdot\cos\gamma.$

25.* $\sin^2\alpha+\sin^2\beta-\sin^2\gamma = 2\sin\alpha\cdot\sin\beta\cdot\cos\gamma.$

26.* $\cos\alpha+\cos\beta+\cos\gamma = 1+4\sin\dfrac{\alpha}{2}\cdot\sin\dfrac{\beta}{2}\cdot\sin\dfrac{\gamma}{2}.$

27.* $\cos\alpha+\cos\beta-\cos\gamma = 4\cos\dfrac{\alpha}{2}\cdot\cos\dfrac{\beta}{2}\cdot\sin\dfrac{\gamma}{2}-1.$

28.* $\cos 2\alpha+\cos 2\beta+\cos 2\gamma = -1-4\cos\alpha\cdot\cos\beta\cdot\cos\gamma.$

29.* $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1-2\cos\alpha\cdot\cos\beta\cdot\cos\gamma.$

30.* $\operatorname{tg}\alpha+\operatorname{tg}\beta+\operatorname{tg}\gamma = \operatorname{tg}\alpha\cdot\operatorname{tg}\beta\cdot\operatorname{tg}\gamma.$

31.* $\operatorname{ctg}\alpha+\operatorname{ctg}\beta+\operatorname{ctg}\gamma = \operatorname{ctg}\alpha\cdot\operatorname{ctg}\beta\cdot\operatorname{ctg}\gamma.$

32.* $\operatorname{tg}\frac{\alpha}{2}\cdot\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\beta}{2}\cdot\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\gamma}{2}\cdot\operatorname{tg}\frac{\alpha}{2}=1.$

33.* Tregoni se nëse për brinjët e trekëndëshit α, β, γ vlen

$$\operatorname{tg}(\alpha-\beta)+\operatorname{tg}(\beta-\gamma)+\operatorname{tg}(\gamma-\alpha)=0.$$

atëherë ai trekëndësh është barabrinjës.

34.* Vërtetoni se nëse $\sin\alpha+\sin\beta=2\sin(\alpha+\beta)$, ku $\alpha+\beta\neq k\pi$, atëherë $\operatorname{tg}\frac{\alpha}{2}\cdot\operatorname{tg}\frac{\beta}{2}=\frac{1}{3}.$

35.* Vërtetoni se

$$\sin\alpha+\sin\beta+\sin\gamma+\sin\delta=4\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\beta+\gamma}{2}\cdot\sin\frac{\gamma+\alpha}{2}$$

ku $\alpha+\beta+\gamma+\delta=2\pi.$

36.* Nëse $\operatorname{tg}\frac{\theta}{2}=\operatorname{cosec}\theta-\sin\theta$, tregoni se $\cos^2\frac{\theta}{2}=\cos 36^\circ.$

37.* Nëse $\operatorname{tg}\theta=\frac{\sin a-\cos a}{\sin a+\cos a}$, tregoni se $\sqrt{2}\sin\theta=\sin a-\cos a.$

38.* Caktoni këndin α nëse:

a) $\operatorname{tg}\alpha=\sqrt{6}+\sqrt{3}-\sqrt{2}-2$; b) $\operatorname{ctg}\alpha=2+\sqrt{2}+\sqrt{3}+\sqrt{6}.$

39.* Të caktohet $\frac{\sin a+\cos a}{\sin a-\cos a}$, nëse

$$\sin 2a=m,\ m\in(-1{,}0)\ \text{dhe}\ 2a\in\left(\frac{3\pi}{2},2\pi\right).$$

40.* Nëse $\cos\alpha=\cos\beta\cdot\cos\gamma$ dhe $\cos\alpha+\cos\beta=0$, tregoni se

$$\operatorname{tg}\frac{\alpha+\beta}{2}\cdot\operatorname{tg}\frac{\alpha-\beta}{2}=\operatorname{tg}^2\frac{\gamma}{2}.$$

41.* Vërtetoni se

$$\cos^2\alpha+\cos^2 2\alpha+...+\cos^2 n\alpha=\frac{n}{2}+\frac{\sin n\alpha\cdot\cos(n+1)\cdot\alpha}{2\sin\alpha}.$$

42.* Është dhënë

$$a_1\cos a_1+a_2\cos a_2+...+a_n\cos a_n=0,$$

$$a_1\cos(a_1+1)+a_2\cos(a_2+1)+...+a_n\cos(a_n+1)=0\,.$$

Tregoni se për çdo b vlen

$$a_1\cos(a_1+b)+a_2\cos(a_2+b)+...+a_n\cos(a_n+b)=0.$$

43.* Nëse $n\equiv 1\ (\mathrm{mod}\,5)$ dhe $n\equiv 2\ (\mathrm{mod}\,5)$ $(n\in N)$ vërtetoni se vlen

$$\cos\frac{2n-1}{5}\pi+\cos\frac{4n}{5}\pi=0\,.$$

44.* Nëse numri n nuk e pjesëton numrin 7, vërtetoni se

$$E=\cos\frac{n\pi}{7}+\cos\frac{3n\pi}{7}+\cos\frac{5n\pi}{7}=\begin{cases}\frac{1}{2} & (n-\text{tek})\\ -\frac{1}{2} & (n-\text{çift})\end{cases}.$$

Të analizohet rasti kur $n = 7k$, $k \in Z$.